(b) Using Floquet states to evolve a quantum system in time

The Floquet states of a time-periodic potential $V(x,t) = V(x,t+\tau)$ play a role analogous to the energy-eigenstates (of the Hamiltonian) of a time-independent system. Once we have the Floquet states, we can evolve an arbitrary initial state in time. Suppose we have calculated the set of Floquet states for a potential $V(x,t) = A(x)\sin\Omega t$.

If we have initial quantum state $\Psi(x,t=0)$ which we wish to evolve to a later time t, under the action of the time-periodic potential $V(x,t)$, we:

1. Expand this state as a superposition of Floquet states at $t=0$.

$$\Psi(x,t=0) = \sum_l d_l \psi_l^F(x,t=0) \tag{1}$$

   Since the Floquet states are orthonormal (compare with Eq.1.22), the coefficients of the superposition are:

$$d_l = <\psi_l^F(x,t=0)\Psi(x,t=0)> = \int_{-\infty}^{+\infty} \psi_l^F(x,t=0)^*\Psi(x,t=0)dx \tag{2}$$

2. Then the quantum state, at later time t is given by:

$$\Psi(x,t) = \sum_l d_l \psi_l^F(x,t) = \sum_l d_l U_l(x,t) e^{-i\epsilon_l t} \tag{3}$$

   Note that $U_l(x,t=0) = U_l(x,t=N\tau)$ where $N\tau$ is an integer multiple of the period. So if we are only looking at our quantum state intermittently (once per period, at $t = 0, \tau, 2\tau ... N\tau$ - this is usually good enough, since a wavepacket is gradually evolving over many periods of the typical oscillation frequency of a laser,say), we can write:

$$\Psi(x,t) = \sum_l \; d_l \psi_l^F(x,t=0) \; e^{-i\epsilon_l t} \tag{4}$$

This looks remarkably like the time evolution of a quantum state in a time-independent potential (Eqs 1.20-1.22 in the lecture notes). But in the latter, the eigenstates are energy-eigenstates (ie eigenstates of H, not of $T(\tau,0)$). The phase ϵ_n above, is sometimes called the 'quasi-energy'. In effect we have reduced the time evolution of a quantum state in a time-dependent potential (which usually requires the solution of the full TDSE) to a form analogous to a time-independent system , by solving an eigenvalue equation (eq.1.54 in notes) similar to the TISE.

We did pay some price for this 'reduction' though: the eigenvalue equation we have to solve to get the Floquet states looks like that of a 2-dimensional system (x and t are the two coordinates). Our matrix with elements $< mnFm'n' >$ has a correspondingly larger dimension. eg if we cut-off our basis states at some maximum value, eg $m = 0,1,2,3...M-1$ and $n = 0,1,2,3,...N-1$, the size of our matrix is $MNXMN$. In contrast, a 1-D system (as in Problem 3 in Homework 1) with a matrix $< nHn' >$ would have a matrix of size NXN.

How to calculate Floquet states

Although the Floquet states are the eigenstates of $T(\tau,0)$, we usually calculate them from another eigenvalue equation

$$FU_n(x,t) = \hbar\epsilon_n U_n(x,t) \tag{5}$$

(Eq.1.54- see derivation in notes). The $U_n(x,t) = U_n(x,t+\tau)$ and then $\psi_n^F(x,t) = exp{-}i\epsilon_n t U_n(x,t)$ from the Floquet Theorem.

The eigenvalue equation (5) is usually solved in its matrix form (as explained in Appendix I.6). We expand $U_n(x,t)$ in a complete set (a 'basis') of known functions of x and t :

$$U_n(x,t) = \sum_{m,k} C_{mk}^n \phi_k(x) \exp im\Omega t = \sum_{m,k} C_{mk}^n |mk> \tag{6}$$

This is equation (1.55) in the notes. The C_{mk}^n are unknowns we need to calculate to specify in full the $n-th$ Floquet state. We use the basis states to generate a matrix equation:

$$\mathbf{F}\underline{\mathbf{C}}^{(n)} = \hbar\epsilon_n \underline{\mathbf{C}}^{(n)} \tag{7}$$

This is Eq. (1.57) in the notes. $\mathbf{F}$ is a matrix with elements $< mk\hat{F}m'k' >$, while $\underline{\mathbf{C}}^{(n)}$ is a column vector with elements C_{km}^n which corresponds to eigenvalue ϵ_n . Once we know the eigenvalues (ie we know the ϵ_n) and the eigenvectors (the coefficients C_{km}^n), then we know the form of the Floquet states of the system. The eigenvalues and eigenvectors are obtained with a computer. In the homework you were just asked to give a formula for the matrix elements of F.

We want to evaluate the matrix elements of the Floquet operator $< mk|F|m'k' >$. From above we know, $|mk >= \phi_k(x) \exp im\Omega t$.

In Homework 2 problem 3: since our oscillating potential had no x dependence we had the (rather physically trivial and unrealistic) simplification $\phi_k(x) = \phi_n(x)\delta_{kn}$ ie one harmonic oscillator eigenstate per Floquet state, hence $C_{km}^n \equiv C_m^n$. It does makes the matrix elements which are $< mn|F|m'n' >$ a bit simpler in our example. Here, $|mn >= \phi_n(x) \exp im\Omega t$. The eigenfunctions of H_0, the $\phi_n(x)$ were, in that example, the Harmonic Oscillator energy eigenstates $H_0\phi_n(x) = (n+1/2)\hbar\omega\phi_n(x)$. If we know the coefficients C_m^n and the phases ϵ_n we know the Floquet states of this system, and we obtained them by diagonalising the matrix of F. We evaluated $< mnFm'n' >=< mn|H_0 + A(x)\sin\Omega t - i\hbar\delta/\delta t|m'n' >$.

Taking things beyond the lecture notes (so you don't have to learn this!), and to clarify the roles of the coefficients, we can return to Eq(3) and substitute the form (6) we have calculated for our functions $U_l(x,t)$:

$$\Psi(x,t) = \sum_l d_l U_l(x,t) e^{-i\epsilon_l t} = \sum_l d_l e^{-i\epsilon_l t} \sum_{m,k} C_{mk}^l \phi_k(x) \exp im\Omega t \tag{8}$$

$$= \sum_l d_l e^{-i\epsilon_l t} \sum_{m,k} C_{mk}^l \ \phi_k(x) \exp im\Omega t = \sum_{l,k,m} d_l \ C_{mk}^l \ \phi_k(x) e^{i(m\Omega-\epsilon_l)t} \tag{9}$$

which, in our simplified example, since $C_{mk}^l \equiv C_m^l$,

$$= \sum_l d_l e^{-i\epsilon_l t} \sum_m C_m^l \phi_l(x) \exp im\Omega t = \sum_{l,m} d_l \ C_m^l e^{-i(\epsilon_l - m\Omega)t} \phi_l(x) \tag{10}$$

The above, should hopefully clarify the distinction between the two sets of coefficients: the d_l which are used to expand the evolving quantum state as a superposition of Floquet states; and the C^l_{mk}, which are used to calculate the individual Floquet states in terms of a superposition of known functions of x and t. In the lecture, the d_l were given as a script-style c_l, but were copied down as an ordinary C_m by some- hence could easily become confused with the C^l_{mk}.