MSci 4261 ELECTROMAGNETISM

LECTURE NOTES I

1.1 Historical Background

This course will deal with electromagnetism. This is currently understood as part of the electroweak theory, unifying the weak nuclear and electromagnetic forces, which in turn is part of the Standard Model which also includes the strong nuclear force mediated by quarks. This model has so far been extremely successful in predicting phenomena up to energies of more than 100GeV. The most successful part of the Standard Model, and the most successful theory in history, is however Quantum Electrodynamics. Its prediction of the magnetic moment of the electron, for example, has been proved correct to at least twelve significant figures (writing the gyromagnetic ratio g in terms of the quantity a via $g = 2(1 + a)$, the experimental and theoretical values are $a_{exp} = .001159652188...$ and $a_{theory} = .001159652188...$, where there are uncertainties only in the last digit of each [1]).

The history of the development of the understanding of electric and magnetic phenomena is intimately linked with the development of physics, and indeed science [2]. The ancient Greeks were already aware of the properties of amber ($\eta\lambda\epsilon\kappa\tau\rho\sigma\nu$) - that when rubbed it attracts light bodies. They also knew that magnetic iron ore ($\eta\ \lambda\iota\theta\sigma\zeta\ \mu\alpha\gamma\nu\eta\tau\iota\zeta$) attracted iron.

The first comprehensive study of electricity and magnetism was presented in 1600 by William Gilbert, a physician to Queen Elizabeth. He proposed that the earth is a magnet, explaining the properties of compasses (which had been known since at least the 12th century). Gilbert found that friction causes forces in many substances other than amber, including glass, sulphur, sealing wax and precious stones. He called this an *electric* force. He further noted the differences between electricity and magnetism. Electricity was caused by friction, seemingly attracted everything, was blocked by screens or water, and caused no definite patterns in attracting other objects. In contrast, magnetism was not affected by friction, only attracted certain other materials, was not blocked by screens or water, and also created definite patterns of attraction (eg using iron filings). Gilbert was familiar with the *humours* of medicine (derived from the Greeks) - phlegm, blood, choler and melancholy - and of the theory of how the effluvia of such humours caused medical conditions. He proposed a similar explanation of electric force as being due to the effluvium of a humour - an influential idea of fluid flow being relevant to electricity. Note that this also contains the important idea of there being no action at a distance.

By the 18th century it was found that there were two forms of electricity - called vitreous (eg produced by glass) and resinous (eg from amber). This was explained in terms of a superfluity or deficiency of electric fluid, together with the crucial idea of conservation of fluid (charge) in any transference of electric substance. Already by 1766 it was found that the electric force obeys an inverse square law. This was argued by Priestley, who noted that for an electrified hollow metallic vessel there was no charge on the inside of the vessel, nor any force acting inside. This was precisely like the then known behaviour of gravity. (The analogy with gravity also led to the proposal of an electric potential ϕ, related to the electric field $\mathbf{E}$ by $\mathbf{E} = -\boldsymbol{\nabla}\phi$, with $\boldsymbol{\nabla}^2\phi$ related to the charge density.)

It was the 19th century which saw the development of classical electrodynamics as we now know it. This was due to the work of many people; the two contributors that I will mention here are Faraday and Maxwell. In 1812 Michael Faraday was a 21 year old bookbinder's assistant. He used to read the books which he was binding, and attend popular lectures. Prompted by some of these books and lectures, he wrote to Sir Humphrey Davy at the Royal Institution, expressing interest in modern developments in electricity and magnetism, and hoping to get any sort of work there. Davy took him on, and to cut a long story short, by 1829 Faraday had succeeded him as Director. Faraday had the idea of *lines of magnetic force*, filling space around a magnet, with the direction of the force along the lines and the intensity of the force increasing with the number of lines per unit area perpendicular to the lines. The magnetic field strength $\mathbf{B}$ was to be thought of as the velocity of an incompressible fluid. Faraday discovered that a changing magnetic field would induce a current in a wire - thus linking electricity and magnetism. Maxwell, born in 1831, was inspired by Faraday's work and ideas. These led him to formulate a unified theory of electricity and magnetism which was encapsulated in his celebrated equations.

1.2 Units and Fundamental Constants

Any more than cursory consideration of the question of the definition of units will reveal that this is a subtle issue, and we will make only very brief comments here. In this course we will mostly use the SI system of units (see an Appendix of Jackson for a fuller discussion of units). In this system, the unit of mass M is the kilogram, which is defined to be the mass of a certain metal bar in Sevres, France. The unit of time T is the second, defined in terms of a certain number of caesium electron transition cycles. Length L is defined in units of the metre, which is defined as the distance travelled by light in 1/299792458 th of a second. Notice that hence *by definition* the speed of light is *exactly* $c = 299792458$ metres per second. Finally, current I is defined in units of the (absolute) ampere, an ampere being that current in two parallel infinite conductors one metre apart which generates 2×10^{-7} Newton metres^{-1} of force between the wires.

It is further convenient to *define* the constants

$$\mu_0 = 4\pi \times 10^{-7} MLT^{-2}I^{-1}$$

and

$$\epsilon_0 = \frac{1}{\mu_0 c^2} = 8.854 \times 10^{-12} ML^3T^{-4}I^{-2}.$$

The cgs system of units is also used in some other sections of Jackson.

As far as fundamental constants are concerned, apart from the fixed quantities c, μ_0, ϵ_0 defined above, we will use Planck's constant $h = 6.626 \times 10^{-34}$ J.sec, the electron charge $e = 1.60 \times 10^{-19}$ Coulombs, and the electron mass $m_e = 9.11 \times 10^{-31}$kg.

1.3 A Reminder of Vector Calculus

To formulate Maxwell's equations most succintly, we will need the language of vector calculus. We will use rectilinear coordinates (x, y, z), and partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. A *scalar field* $\phi(x, y, z)$ is a function of the coordinates (and possibly also of time t). A *vector field* $\mathbf{V}$ is a vector whose components are functions, so that $\mathbf{V} = (V_x, V_y, V_z)$, with each of the components V_x, V_y and V_z being functions of x, y and z (and possibly t). The vector of derivatives $\boldsymbol{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, called *del* or *nabla*, will play an essential role. From a scalar field ϕ we may form a vector field given by

$$\mathrm{grad}\,\phi = \boldsymbol{\nabla}\phi = (\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}).$$

From a vector field $\mathbf{V}$ we may form the scalar field

$$\mathrm{div}\mathbf{V} = \boldsymbol{\nabla} \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

Finally, from the vector field $\mathbf{V}$, we may form another vector field

$$\mathrm{curl}\,\mathbf{V} = \boldsymbol{\nabla} \times \mathbf{V} = (\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}).$$

Some identities are

$$\begin{aligned}
\mathrm{div}\,\mathrm{curl}\,\mathbf{V} &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{V}) = 0, \\
\mathrm{curl}\,\mathrm{grad}\,\phi &= \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0, \\
\boldsymbol{\nabla} \cdot (\phi\mathbf{V}) &= \mathbf{V} \cdot \boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla} \cdot \mathbf{V}, \\
\boldsymbol{\nabla} \times (\phi\mathbf{V}) &= (\boldsymbol{\nabla}\phi) \times \mathbf{V} + \phi(\boldsymbol{\nabla} \times \mathbf{V}), \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{V}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \boldsymbol{\nabla}^2\mathbf{V}
\end{aligned}$$

(recall also that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ for any vectors, or vector fields, $\mathbf{a}, \mathbf{b}, \mathbf{c}$).

There are also some important relations involving integrals. Consider first a closed region of space V surrounded by a boundary surface S. Let $\mathbf{n}$ be a unit vector normal to the boundary surface, dS the infinitesimal area element on the boundary (with $d\mathbf{S} = \mathbf{n} dS$), and $\mathbf{A}$ a vector field. Then we have *Gauss' law*:

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \oint_S \mathbf{A} \cdot \mathbf{n}\, da = \int_V \boldsymbol{\nabla} \cdot \mathbf{A}\, d^3x.$$

Now consider a different surface S which has a boundary loop C. If $d\mathbf{l}$ is the infinitesimal line element along C, then we have *Stokes' theorem*:

$$\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{A} \cdot d\mathbf{l}.$$

Further identities and theorems can be found in the textbooks, for example inside the front cover of Jackson.

The above two equations have simple interpretations in terms of the geometry of vector fields - for the first, if $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ in some region V with boundary S, then $\oint_S \mathbf{A} \cdot d\mathbf{S} = 0$, which means that the flux of the vector field $\mathbf{A}$ into the surface S is equal to the flux out of the surface, or in other words, that there are no *sources* of flux in the region V. As for the second equation, from Stokes theorem, one sees that if a vector field $\mathbf{A}$ satisfies $\boldsymbol{\nabla} \times \mathbf{A} = 0$ on a surface S with boundary curve γ, then $\oint_\gamma \mathbf{A} \cdot d\mathbf{l} = 0$, which means that there are no *loops* of flux of $\mathbf{A}$ - it is 'irrotational'. ($d\mathbf{l} = \mathbf{t} dl$, where $\mathbf{t}$ is the unit tangent vector to the curve and dl is the infinitesimal length element.)

To motivate the introduction of Maxwell's equations below, let us note that, modulo some subtleties, you can fully specify a vector field $\mathbf{A}$ by giving $\boldsymbol{\nabla} \cdot \mathbf{A}$ and $\boldsymbol{\nabla} \times \mathbf{A}$ (see Jackson, Chapter 6, page 241 for a discussion). To see this, firstly note that you can write $\mathbf{A} = \mathbf{A}_d + \mathbf{A}_c$, with $\boldsymbol{\nabla} \cdot \mathbf{A}_d = 0$ and $\boldsymbol{\nabla} \times \mathbf{A}_c = 0$. (This is true up to the addition of a term $\boldsymbol{\nabla}\Phi$ with $\boldsymbol{\nabla}^2\Phi = 0$, but for suitable regions and boundary conditions this has a unique solution.) This result is a special case of a general result, the Hodge decomposition theorem, which splits a differential form into closed, co-closed and harmonic pieces - see, for example, M. Nakahara, *Geometry, Topology and Physics*, Institute of Physics Publishing 1990, Section 7.9.

Then, equivalently, given $\boldsymbol{\nabla} \cdot \mathbf{A}$ you can find $\mathbf{A}_c$, and given $\boldsymbol{\nabla} \times \mathbf{A}$ you can find $\mathbf{A}_d$ (the integration constants involved being fixed by suitable boundary conditions). Thus, if we know the div and curl of a vector field, then it is fully specified - so to determine the electric and magnetic fields we need to find the right equations for their div and curl- and these four equations are precisely Maxwell's equations.

1.4 Maxwell's Equations

Thus we turn to Maxwell's equations, which present a unified description of electricity and magnetism.

1.4.1 In Vacua

We saw above that the analogy between electrical force and gravitational force led to the introduction of an *electric potential* ϕ, related to the electric field $\mathbf{E}$ by $\mathbf{E} = -\boldsymbol{\nabla}\phi$. The electric field gives the force on a unit charge, and one can find the spatial surfaces where the magnitude of the force is the same - these are exactly the surfaces where ϕ is constant - *equipotential* surfaces.

Following this gravitational analogue, $\boldsymbol{\nabla} \cdot \mathbf{E} = -\boldsymbol{\nabla}^2\phi$ must be related to the charge density ρ, which is zero in a vacuum. This leads to the first of Maxwell's equations in a vacuum:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0.$$

The magnetic analogue of the above equation is

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0,$$

where $\mathbf{B}$ is the magnetic field. Faraday found that a changing magnetic field induced an electric field, according to his *Law of Induction*:

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

The magnetic equivalent of this is

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

(The term involving $\frac{\partial \mathbf{E}}{\partial t}$ was introduced by Maxwell and called the *displacement current*. Given the law of induction, the equation above is a consequence of special relativity, as we will see later.) We have introduced a constant, expressed as $\frac{1}{c^2}$, in the above equation. In 1849, Kohlrausch and Weber found experimentally that the constant c was equal to 3.1×10^{10} cm/sec.

We will keep the equations $\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \boldsymbol{\nabla} \cdot \mathbf{B} = 0$ in the non-static case, where the fields may vary with time. Thus, Maxwell's equations in a vacuum are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t},$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Note that these equations are consistent with the vector calculus identity $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{V} = 0$, for any vector field $\mathbf{V}$ – as one can see by taking the divergence of the second and fourth equations and using the first and third.

Taking the curl of the equations involving the curls of $\mathbf{E}$ and $\mathbf{B}$, one finds that

$$\boldsymbol{\nabla}^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2},$$
$$\boldsymbol{\nabla}^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

These are *wave equations* - for example, $\mathbf{E} = (\cos(x - ct), 0, 0)$ is a solution of the first equation. The wave velocity is given by the constant c, which we saw was known at that time to be a number near to 3.1×10^{10} cm/sec. The speed of light had also been measured during this period - for example, Fizeau in 1849 found the value 3.15×10^{10} cm/sec, using an apparatus with a spinning toothed wheel and mirror.

Thus, the constant in the wave equations of electromagnetic theory was found to be very close to the speed of light. Maxwell described the next step: *"We can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena"*. This conclusion was one of the great achievements of nineteenth century science.

1.4.2 Sources

Now we turn to the effects of *sources* of the fields. Consider first the introduction of electrostatic sources, *ie*, an electric charge distribution which is constant in time. If the charge is distributed according to a charge density $\rho(\mathbf{r})$, then the first of Maxwell's equations becomes

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho,$$

for a constant ϵ_0 which will be discussed shortly. The corresponding potential Φ, with $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, satisfies

$$\boldsymbol{\nabla}^2 \Phi = -\frac{1}{\epsilon_0}\rho$$

(this is exactly analogous to Newtonian gravity). We note that a solution to this equation is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV,$$

where the integral is over all space. To see this, note that (with $r = |\mathbf{r}|$)

$$\boldsymbol{\nabla}^2 \frac{1}{r} = -4\pi\delta^3(\mathbf{r}) \qquad (*).$$

A not very precise proof is by the following argument - firstly, when $|\mathbf{r}| \neq \mathbf{0}$, one can prove the above directly by simple computation. Secondly, if one integrates the left-hand side of the above equation over a unit ball B, with bounding two sphere S^2, then one finds that

$$\int_B (\boldsymbol{\nabla}^2 \frac{1}{r})\, d^3x = \int_{S^2} \boldsymbol{\nabla}(1/r) \cdot d\mathbf{S} = -\int_{S^2} \frac{1}{r^3}\, \mathbf{r} \cdot d\mathbf{S} = -\int_{r=1} \frac{r^2}{r^4}\, d\Omega = -4\pi$$

($d\Omega$ is the unit solid angle). The two properties of the left-hand side of equation (*) just proved are those shared by (and partially defining) the function on the right-hand side of that equation. (A more complete argument proving this relation can be given.)

For a point charge Q, located at the origin $\mathbf{r} = (0,0,0)$, the charge density is given by $\rho(\mathbf{r}) = Q\delta(\mathbf{r})$, and from the properties of the delta function we have (with $r = |\mathbf{r}|$)

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}, \quad \mathbf{E} = -\boldsymbol{\nabla}\Phi = \frac{1}{4\pi\epsilon_0}\frac{Q\mathbf{r}}{r^3}.$$

In our SI units, $\epsilon_0 = 8.854 \times 10^{-12} C^2 N^{-1} m^{-2}$, and is called the *permittivity of free space.*

Since there are no magnetic charges (see Section 1.5 below), we conclude that there are no source terms to add to the equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. However, electric currents do generate magnetic fields. An important advance in understanding this was made by Biot and Savart in 1820. For a constant current I, flowing in a straight wire oriented along the z axis, the following magnetic field was found to be generated:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{2\pi}\frac{I}{\rho}\hat{\phi},$$

where we are using the standard cylindrical coordinates, with ρ the perpendicular distance to the z axis, and ϕ the radial coordinate around the z axis, with unit coordinate vector $\hat{\phi}$. We have used $\mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi}\frac{1}{\rho}\hat{\phi} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} d\phi$.

The constant μ_0, called the *permeability of free space* is given by $1.257 \times 10^{-6} N Amp^{-2}$. Integrating around a closed path γ which encloses the wire, we find

$$\int_S \boldsymbol{\nabla} \times \mathbf{B} \cdot d\mathbf{S} = \oint_\gamma \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0}{2\pi} I \int d\phi = \mu_0 I = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S},$$

where S is any surface whose boundary is γ, and $\mathbf{J}$ is the electric current density. From the above we can see that we must have $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$ in this case. Finally, since there is no current of magnetic charges, there will be no source term in the $\boldsymbol{\nabla} \times \mathbf{E}$ equation.

This discussion motivates the final presentation of Maxwell's equations in the presence of sources:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{1}{\epsilon_0}\rho, \\
\boldsymbol{\nabla} \times \mathbf{B} &= \epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}, \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}
\end{aligned}$$

(with $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$).

An important consequence of these equations is obtained by taking the divergence of the second equation and using the first, giving

$$\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

This expresses the *conservation of electric charge.* To see this, integrate the above relation over a volume V, with bounding surface S, to find $\int_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \rho$, expressing the equality of the charge leaving the surface S with the change of the amount of charge inside. Another way to motivate the introduction of the current term in the second of Maxwell's equations above is that precisely this term is needed in order to ensure that these equations imply the equation for the conservation of charge.

1.5 Energy and Momentum

Like charge, energy is conserved. Thus, for electromagnetic fields there must be some energy density $\mathcal{E}$ (analogous to electric charge) and energy current $\mathcal{P}$ (analogous to electric current). By conservation, these must satisfy

$$\nabla \cdot \mathbf{P} + \frac{\partial \mathcal{E}}{\partial t} = 0, \quad \text{or} \quad \int_S \mathcal{P} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \mathcal{E}.$$

This equation should follow from Maxwell's equations.

A little investigation yields the following expressions (for the free space case):

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2,$$

$$\mathcal{P} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).$$

The vector $\mathcal{P}$ is known as *Poynting's vector*. This will be discussed further in later lectures. One can similarly define a *momentum* density and current for the electromagnetic field, which will be done in a later lecture.

1.6 Magnetic Monopoles

A homework question asks which modifications of Maxwell's equations would be required if magnetic monopoles existed. A magnetic monopole is the magnetic analogue of an electric charge, and is a source of magnetic field lines - ie if the magnetic charge of a magnetic monopole is g, then the magnetic field produced by a magnetic monopole at the origin is given by

$$\mathbf{B}(\mathbf{r}) = g\frac{\mu_0}{4\pi}\frac{\mathbf{r}}{r^3}.$$

One surprising consequence of the existence of magnetic monopoles is that this would imply a relationship between the electric and magnetic charges of any pair consisting of an electric particle with charge e and a magnetic monopole with charge g. This relationship is

$$ge = \frac{1}{2}n\hbar c$$

where n is an integer. One way to derive this is by considering the *angular momentum* density in the electromagnetic field. This is given by $\mathbf{r} \times \mathbf{p}$ where $\mathbf{p}$ is the momentum density alluded to in the section above. For a set-up where an electric charge and a magnetic monopole lie at different points on the z-axis, one finds that the angular momentum of the electromagnetic field has a z-component equal to $-eg/c$ (see [3] for example). Setting this equal to $n\hbar/2$ for an integer n (using the quantisation of intrinsic angular momentum in quantum mechanics), gives the equation above. Notice that this equation implies that $g^2/e^2 = (1/4)(c\hbar^2/e^2)^2 \simeq (1/4)(137)^2 \simeq 5000$, ie the force between magnetic monopoles is 5000 times stronger than that between electric charges. So far, no magnetic monopoles have been found. A review of experimental limits on monopole masses, and of some theoretical issues, can be found at `http://uk.arXiv.org/abs/hep-ph/0111062`. Limits of at least some hundreds of GeV for monopole masses are argued for in this paper. Magnetic monopoles play an important role in supersymmetric gauge theories and string theory. A review of much of this subject can be found at `http://arxiv.org/abs/hep-th/0609055`.

One can also consider the possibility of *dyons* – particles carrying both electric and magnetic charges. The quantisation condition for these turns out to be

$$e_1 g_2 - e_2 g_1 = \frac{1}{2}n\hbar c.$$

Dyonic particles have proved to be important in recent studies of supersymmetric theories containing electromagnetism, via the study of *duality*. See the last set of lecture notes for some discussion of this.

References

[1] IJR Aitchison and AJG Hey, *Gauge Theories in Particle Physics*, 2nd Edition, Adam Hilger 1989, p. 212.
[2] See E Whittaker, *A History of the Theories of Aether and Electricity* (2 Vols) for an authoritative study.
[3] HC O'Hanian, *Classical Electrodynamics*, Allyn and Bacon 1988, Section 9.7.