MSci 4261 ELECTROMAGNETISM

LECTURE NOTES II

2.1 Electricity, Magnetism and Matter

In the previous notes, the equations governing electromagnetism in a vacuum, with source terms, were discussed. These equations are just four in number, describing the div and curl of the electric and magnetic fields. We will have more to say about these equations later in the course. At this point we turn to the consideration of electromagnetic fields in matter. This is in general a very complex situation, with matter exhibiting a variety of electric and magnetic behaviour, such as dielectricity, ferroelectricity, piezoelectricity, paramagnetism, diamagnetism, ferromagnetism, superconductivity, etc. However, under conditions where the fields are not strong and with matter which is not in an unusual state, one can simplify the discussion considerably.

2.2 Polarisation and the Electric Displacement Field

Matter which is not a perfect conductor is able to support an electric field in its interior without flow of current. The field however acts on the charged nuclei and electrons, causing *polarisation.* A simple model of this involves atoms at fixed sites acquiring electric dipole moments. We can define the *polarisation field* $\mathbf{P}(\mathbf{r})$ of the insulator to be such that the dipole moment $\mathbf{d}$ in any finite volume V is given by

$$\mathbf{d} = \int_V \mathbf{P} dV = -\int_V \mathbf{r}\ (\boldsymbol{\nabla}\cdot\mathbf{P})\ dV + \int_S \mathbf{r}\ (\mathbf{P}\cdot d\mathbf{S})$$

(one can check the second equality for each vector component using integration by parts). Physically then $\mathbf{d}$ is the electric dipole moment per unit volume. If we now choose the volume V to enclose all matter, with the boundary of V *outside* all matter, then $\mathbf{P}$ vanishes on the bounding surface -

$$\mathbf{P}|_{\mathbf{S}} = \mathbf{0},$$

and thus

$$\mathbf{d} = -\int_V \mathbf{r}(\boldsymbol{\nabla}\cdot\mathbf{P})\ dV.$$

However, by definition the dipole moment is given by the integral $\mathbf{d} = \int \mathbf{r}\rho_b(\mathbf{r})dV$, where ρ_b is the bound charge, ie in the matter sample. Thus we conclude that

$$\boldsymbol{\nabla}\cdot\mathbf{P} = -\rho_{\mathbf{b}}.$$

Thus the macroscopic density of bound charge ρ_b giving the polarisation is given by $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$. There may also be an additional 'free' charge density ρ_f. Maxwell's equation for $\mathbf{E}$ is then

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\epsilon_0}(\rho_b + \rho_f) = \frac{1}{\epsilon_0}(-\boldsymbol{\nabla}\cdot\mathbf{P} + \rho_f).$$

Defining the new *electric displacement field* $\mathbf{D}$ by

$$\mathbf{D}(\mathbf{r}) = \epsilon_{\mathbf{0}}\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$$

we have the equation

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_f.$$

If the external field $\mathbf{E}$ is not too large to break down the structure of the insulator, then typically it is found that the polarisation field induced is proportional to the electric field, ie

$$\mathbf{P}(\mathbf{r}) = \epsilon_0\chi_e\mathbf{E}(\mathbf{r})$$

where the constant χ_e is called the *electric susceptibility.* (We are assuming that the matter is uniform and isotropic.) Insulators of this type are called *dielectrics.* Thus, for dielectrics,

$$\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$$

where $\epsilon_r = 1 + \chi_e$ is called the *relative permittivity* or *dielectric constant.*

2.3 Magnetisation and the H Field

There is an analogous discussion to that above for magnetic fields. Matter placed in an external magnetic field will react, for instance via diamagnetism, where opposing atomic magnetic moments are induced by the external magnetic field.

Here we will again assume that there can be defined a smooth averaged macroscopic field, here called the *magnetisation field* $\mathbf{M}(\mathbf{r})$ such that the total magnetic moment $\mathbf{m}$ in a volume V is given by

$$\mathbf{m} = \int_V \mathbf{M}(\mathbf{r})\ dV,$$

so that $\mathbf{m}$ is the magnetic moment per unit volume. First note the vector identity

$$\mathbf{r} \times (\boldsymbol{\nabla} \times \mathbf{M}) = 2\mathbf{M} + \boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{M}) - (\boldsymbol{\nabla} \cdot \mathbf{r})\mathbf{M} - (\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{M}$$

(this is straightforwardly proved by considering each component separately). Then

$$\int_V \mathbf{M}\ dV = \int_V \left(\frac{1}{2}\mathbf{r} \times (\boldsymbol{\nabla} \times \mathbf{M}) - \boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{M}) + (\boldsymbol{\nabla} \cdot \mathbf{r})\mathbf{M} + (\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{M}\right) dV$$

The second term on the right-hand side of the above equation gives zero - as each component gives a total derivative, which integrates to a boundary integral, which is zero since we choose V so that there is no matter at or outside the boundary and hence $\mathbf{M} = \mathbf{0}$ there. A similar argument applies when one combines the last two terms on the right-hand side of the above equation; again this is most easily seen for each component separately.

Thus

$$\mathbf{m} = \frac{1}{2}\int_V \left(\mathbf{r} \times (\boldsymbol{\nabla} \times \mathbf{M})\right)\ dV.$$

However, by definition the magnetic moment is given by

$$\mathbf{m} = \frac{1}{2}\int_V \left(\mathbf{r} \times \mathbf{J}_b\right)\ dV$$

where $\mathbf{J}$ is the current. Thus we can identify the current distribution corresponding to the magnetisation as

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M}.$$

If there are additional free currents $\mathbf{J}_f$, then Maxwell's equation for $\boldsymbol{\nabla} \times \mathbf{B}$ becomes

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{J}_f + \mathbf{J}_b) = \mu_0(\mathbf{J}_f + \boldsymbol{\nabla} \times \mathbf{M}).$$

Define the field $\mathbf{H}$ by

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

Then we have

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f.$$

For many materials it is found experimentally that the magnetisation induced by an external magnetic field is proportional to that external field. Thus, for uniform isotropic matter of this type

$$\mathbf{M} = \chi_m \mathbf{H}$$

where the constant χ_m is called the *magnetic susceptibility* of the material. From the above equations we have

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$

where $\mu_r = 1 + \chi_m$ is the *relative permeability.*

In this and the previous section, various assumptions were made which should be kept in mind as they do not always apply. For example, the matter under consideration may not be uniform or isotropic. The assumption that the matter responds *linearly* to the applied external fields also is not tenable under certain circumstances - for example in the presence of strong fields. In these circumstances one must consider non-linear responses, eg depending upon the square of the external field. Finally, the averaged higher moments (eg quadrupole) have been ignored in the above discussions.

2.4 Maxwell's Equations in Matter

Including the time dependence of fields, we are led by the above considerations to Maxwell's equations in matter: (recall that $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}, \mathbf{H} = \frac{1}{\mu_0\mu_r}\mathbf{B}$)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{D} &= \rho,\\
\boldsymbol{\nabla}\times\mathbf{H} &= \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t},\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0,\\
\boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}.
\end{aligned}$$

Note that one describes the type of matter discussed in the previous two sections as *linear media*, due to the linear relations between the polarisation and the applied electric field, and the magnetisation and the applied magnetic field.

The additional law we will need describes the motion of particles of charge q moving in external electromagnetic fields $\mathbf{E}, \mathbf{B}$. This is the *Lorentz force law*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

The electric field $\mathbf{E}$ is given by the force $\mathbf{F}$ on a unit charge. The work done in moving a charge q a distance $d\mathbf{x}$ is $dW = \mathbf{F}\cdot d\mathbf{x} = q\mathbf{E}\cdot d\mathbf{x}$, so that the rate of change of energy is given by

$$\frac{dW}{dt} = q\mathbf{v}\cdot\mathbf{E}$$

with $\mathbf{v}$ the velocity. Note that the magnetic field $\mathbf{B}$ does no work as the force due to $\mathbf{B}$ acts perpendicularly to the direction of motion (this follows immediately from the Lorentz force law since $\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = \mathbf{0}$ identically). Since the magnetic field does no work it does not change the energy of a particle.

2.5 Boundary Conditions at Matter Interfaces

Maxwell's equations imply constraints on the electric and magnetic fields at interfaces between two media. In the following we will derive relations which can be used in solving the Maxwell equations in different regions and connecting the solutions via continuity relations.

Firstly, write Maxwell's equations in integral form as follows. Consider a region of space V, bounded by a surface S. Recall Gauss' law $\int_V \boldsymbol{\nabla}\cdot\mathbf{X}\,dV = \int_S \mathbf{X}\cdot d\mathbf{S}$, for any vector field $\mathbf{X}$. Also, let C be a closed contour in space with line element $d\mathbf{l}$ along the contour, and let S' be an open surface spanning the contour. We also have Stokes' theorem $\oint_C \mathbf{X}\cdot d\mathbf{l} = \int_{S'}(\boldsymbol{\nabla}\times\mathbf{X})\cdot d\mathbf{S}'$. These results may be used to write Maxwell's equations as

$$\begin{aligned}
&\int_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\,dV, \quad \oint_C \mathbf{H}\cdot d\mathbf{l} = \int_{S'}(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t})\cdot d\mathbf{S}'\\
&\int_S \mathbf{B}\cdot d\mathbf{S} = 0, \quad \oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_{S'}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}'
\end{aligned}$$

(these follow by integrating the scalar Maxwell equations over a volume V and the vector equations over a surface S' with element $d\mathbf{S}'$).

Now, suppose two regions, which are labelled by indices $i = 1, 2$, contain different media and contain fields $\mathbf{E}_i, \mathbf{D}_i, \mathbf{B}_i, \mathbf{H}_i$, respectively. Assume the two regions meet at a two-dimensional boundary. Consider first a very small shallow cylinder which straddles the boundary between the two regions, for which the normals to the circular ends of the cylinder are perpendicular to the boundary. Apply the first and third of the Maxwell relations above to the volume and surface of this cylinder. Ignoring the contribution from the infinitely thin sides of the cylinder one finds that $\oint_S \mathbf{D}\cdot d\mathbf{S} = (\mathbf{D}_2 - \mathbf{D}_1)\cdot \mathbf{n}\Delta a$, where Δa is the area of the circular end of the cylinder, and $\mathbf{n}$ the unit normal to the boundary. For the electric case, given a surface charge density σ we have $\int_V \rho\, dV = \sigma \Delta a$. Thus we deduce the boundary conditions

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot \mathbf{n} = \sigma,$$
$$(\mathbf{B}_2 - \mathbf{B}_1)\cdot \mathbf{n} = 0,$$

the second equation following by similar arguments, noting the absence of magnetic charges.

Now consider a small rectangle which straddles the boundary between the two media. This rectangle has short sides which are infinitesimally small, and longer sides of length Δl which are parallel to the boundary. The unit normal $\mathbf{t}$ to the rectangle is tangent to the interface between the regions. Then $\oint_C \mathbf{H}\cdot d\mathbf{l} = (\mathbf{t}\times\mathbf{n})\cdot(\mathbf{H}_2 - \mathbf{H}_1)\Delta l$. Assume that there is a current density $\mathbf{K}$ flowing on the rectangle surface $\mathbf{S}'$. Then $\int_{S'}(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t})\cdot d\mathbf{S}' = \mathbf{K}\cdot\mathbf{t}\Delta l$, since the $\mathbf{D}$ term vanishes as the area of the cylinder goes to zero. Thus we deduce from the second and fourth of the Maxwell integral relations that

$$\mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K},$$
$$\mathbf{n}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0,$$

the second equation following again by the same arguments, noting the absence of magnetic sources.

Thus we see that Maxwell's equations, applied across a boundary between two media, imply that the normal component of $\mathbf{B}$ and the tangential component of $\mathbf{E}$ are continuous across the boundary (from the second equation in each of the two pairs of equations above). Furthermore, the discontinuity of the normal component of $\mathbf{D}$ equals the surface charge density, and the tangential component of $\mathbf{H}$ is discontinuous and equal in magnitude to the surface current density (from the first equation in each pair above).

2.6 Energy and Momentum

We have seen how to define an energy density $\mathcal{E}$ and flow $\mathcal{P}$ for the electromagnetic fields in a vacuum. These satisfy the conservation of energy equation

$$\nabla\cdot\mathcal{P} + \frac{\partial \mathcal{E}}{\partial t} = 0.$$

The vacuum expressions are $\mathcal{E} = \frac{1}{2}\epsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$ and $\mathcal{P} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$.

It is straightforward to generalise these to the case of linear media. One can motivate this in various ways. The quickest approach is to note that whatever the expressions are, they must satisfy the conservation of energy equation as a consequence of Maxwell's equations in matter, and they must reduce to the vacuum expressions when the matter is absent. The other piece of information necessary is that in the presence of currents one has moving charges, and hence work is done on the charges by the electric field (but not by the magnetic field, as we saw above). For a continuous distribution of charge and current, the total rate of doing work by the fields in a volume V is

$$\int_V \mathbf{J}\cdot\mathbf{E}\, dV$$

- for example, if the current is given by

$$\mathbf{J}(\mathbf{r}) = q\dot{\mathbf{x}}(t)\delta(\mathbf{r} - \mathbf{x}(t)),$$

with $\mathbf{x}(t)$ the particle trajectory, then

$$\int \mathbf{J}\cdot\mathbf{E}\; d^3x = q\dot{\mathbf{x}}(t)\cdot\mathbf{E} = \frac{dW}{dt},$$

the rate of work done.

A little consideration then leads one to the formulæ

$$\mathcal{E} = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}),$$
$$\mathcal{P} = \mathbf{E}\times\mathbf{H}.$$

These satisfy the conservation equation

$$\nabla\cdot\mathcal{P} + \frac{\partial\mathcal{E}}{\partial t} = -\mathbf{J}\cdot\mathbf{E}$$

using the Maxwell equations presented above in Section 2.4. Note that these considerations apply to the case of linear media and must be modified in more general situations.

Let us now consider the *momentum* in the electromagnetic field. From the Lorentz force law, considering $\mathbf{P}_m$ to be the mechanical momentum of the particles of the matter, we have

$$\frac{d\mathbf{P}_m}{dt} = \int_V (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\ dV$$

(eg $\frac{d\mathbf{P}_m}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \int_V(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})$ for a continuous distribution of charges and currents).

Now use Maxwell's equations to eliminate ρ and $\mathbf{J}$ from the above equation. After a little algebra, the result is (we take $\epsilon_r = \mu_r = 1$ here)

$$\begin{aligned}\frac{d\mathbf{P}_m}{dt} &+ \frac{d}{dt}\int_V \epsilon_0(\mathbf{E}\times\mathbf{B})dV \\ &= \epsilon_0\int_V \Big(\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E}) + c^2\mathbf{B}(\nabla\cdot\mathbf{B}) - c^2\left(\mathbf{B}\times(\nabla\times\mathbf{B})\right)\Big)dV.\end{aligned}$$

The second term on the left-hand side suggests that the total electromagnetic momentum in the volume V is given by the expression

$$\mathbf{P}_{em} = \frac{1}{c^2}\int(\mathbf{E}\times\mathbf{H})dV.$$

The terms on the right-hand side of the equation representing the time rate of change of momentum must then represent momentum flow in order for this to be the conservation equation.

To see this, define the *Maxwell stress tensor* T_{ab} by

$$\frac{1}{\epsilon_0}T_{ab} = E_aE_b + c^2B_aB_b - \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})\delta_{ab}.$$

We are using the following shorthand notation - indices a, b run over the values $1, 2, 3$, and $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{B} = (B_1, B_2, B_3)$. The identity tensor δ_{ab} is equal to zero if $a \neq b$ and equals one if $a = b$.

The right-hand side of the rate of change of momentum equation can then be written as (it is instructive to show this for eg the $a = 1$ component, and separately for the $\mathbf{E}$ and $\mathbf{B}$ terms, to convince oneself)

$$\int_V \frac{\partial}{\partial x_b}T_{ab}dV = \oint_S T_{ab}n_b dS,$$

where S is the surface bounding the volume V, with unit outward normal vector $\mathbf{n} = (n_1, n_2, n_3)$. We will adopt the *Einstein summation convention*, whereby if an expression contains an index repeated twice, it is assumed that this index is summed over in the expression. An example is the index b in the above equation, where it is assumed that the expression is to be summed with b running from one to three.

Thus we finally come to the result

$$\frac{d}{dt}(\mathbf{P}_m + \mathbf{P}_{em})_a = \oint_S T_{ab}n_b dS.$$

The left-hand side is the a-component of the rate of change of total momentum, including that of the electromagnetic field $\mathbf{P}_{em}$. The right-hand side is the a-component of the flow per unit area of momentum across the surface S into the volume V, ie the force per unit area across the surface, acting on the fields and matter.

Finally, we remark that angular momentum can also be treated in a similar way. The angular momentum density in the electromagnetic field is given by

$$\mathbf{L}_{em} = \mathbf{r} \times \mathbf{P}_{em},$$

where $\mathbf{P}_{em}$ is the momentum in the field discussed above. The flux of angular momentum is given by

$$\mathbf{M} = \mathbf{T} \times \mathbf{r},$$

where $\mathbf{T}$ is the energy-momentum tensor given above (see Jackson, Problem 6.10, for more discussion of this notation). Conservation of angular momentum is expressed by the equation

$$\frac{d}{dt}(\mathbf{L_m} + \mathbf{L}_{em}) + \nabla \cdot \mathbf{M} = 0,$$

where $\mathbf{L_m}$ is the mechanical angular momentum, and this equation follows by use of Maxwell's equations, just as for the equations expressing conservation of energy and momentum which we have discussed earlier.

2.7 The Clausius-Mossotti Relation

As an application of the ideas of polarisation etc to matter, and its relation with microscopic quantities, we now derive the *Clausius-Mossotti relation*, expressing molecular polarisability in terms of the dielectric constant. We consider the matter under consideration to be made from molecules, and under the influence of an applied external macroscopic electric field $\mathbf{E}$, the matter will be polarised. The polarisation of neighbouring molecules will give rise to an internal field $\mathbf{E}_i$ at each molecular site. This internal field will itself be made from the difference of the actual contribution $\mathbf{E}_{near}$ of the molecules close to the given site, and the average contribution from those molecules $\mathbf{E}_P$ due to the overall polarisation

$$\mathbf{E}_i = \mathbf{E}_{near} - \mathbf{E}_P$$

(we will not use the near field here).

Now, it was noted earlier that the potential Φ for a charge distribution ρ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r',$$

where the volume V encloses the distribution. Taking V to be a sphere of radius R, it is possible to show that (see, for example, Jackson, Section 4.1)

$$\int_V \mathbf{E}\, dV = -\frac{1}{3\epsilon_0}\mathbf{p},$$

where $\mathbf{p}$ is the dipole moment of the charge distribution with respect to the center of the sphere. Now, $\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}$, if we take R small enough so that $\mathbf{P}$ is constant inside the sphere. Thus, the average electric field inside the sphere is given by

$$\mathbf{E}_P = \frac{1}{\frac{4}{3}\pi R^3}\int_V \mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}.$$

Whence

$$\mathbf{E}_i = \frac{1}{3\epsilon_0}\mathbf{P} + \mathbf{E}_{near}.$$

We can define the *molecular polarisability* γ_{mol} by the relation $< \mathbf{p_{mol}} >= \epsilon_0 \gamma_{mol}(\mathbf{E} + \mathbf{E}_i)$, where $< \mathbf{p_{mol}} >$ is the average dipole moment of the molecules. Notice that $\mathbf{E} + \mathbf{E}_i$ is the applied field at the molecule. We also have $\mathbf{P} = N\mathbf{p_{mol}}$ where N is the average number of molecules per unit volume. Recall also that $\epsilon = \epsilon_0(1+\chi_e)$. Assuming that the near field vanishes, we thus deduce that

$$\mathbf{P} = N\gamma_{mol}(\epsilon_0\mathbf{E} + \frac{1}{3}\mathbf{P}).$$

Using $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$ we then find that

$$\gamma_{mol} = \frac{3}{N}\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2},$$

which is the Clausius-Mossotti relation, named after the two nineteenth century investigators who established that for any given substance the combination $\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0+2}$ is proportional to the density of the substance. The relation holds best for dilute substances such as gases.

2.8 Appendix: Solved Problems

Here are two solved problems which illustrate the result given for the energy density of the electromagnetic field.

Problem 1:
Consider a parallel plate capacitor with plates of area A and separation x. If the charge on one plate is Q and on the other $-Q$, and the dimensions are such that edge effects may be neglected, what is the magnitude of the electric field E in the region between the plates? What is the force exerted by the one plate on the other? How much work must be done in order to increase the separation of the plates by an amount dx? How much energy is stored in the capacitor when the plates are separated by x? If this energy resides in the electric field, express this energy in terms of the volume of the region in which the electric field exists, and the magnitude of the field there. Hence show that the energy density associated with an electric field is

$$u_E = \frac{1}{2}\epsilon_0 E^2.$$

Answer:
Since we are to neglect edge effects, symmetry arguments may be used to show that the field is everywhere perpendicular to the plates, and is uniform between them. Also the field outside the plates vanishes. So if we consider a "Gaussian cylinder" with end faces parallel to the plates, one being between the plates, the flux of electric field out of the cylinder is just ES, S being the area of the base of the cylinder. But the charge enclosed within the cylinder is QS/A and then Gauss'theorem yields

$$E = \frac{Q}{A\epsilon_0}.$$

This E-field exerts an (attractive) force on the opposite face. There is a subtlety needed to determine the magnitude of this force, which might be thought to be EQ; in fact there is an additional factor $\frac{1}{2}$ which arises because the charge has a field of magnitude E on one side of it and zero on the other. The correct answer, most usually obtained by first determining the energy stored in the capacitor (using an argument based on building up the *charge* from zero to Q, rather than the method we will use below), is

$$F = \frac{1}{2}EQ = \frac{1}{2}A\epsilon_0 E^2.$$

The work done in increasing the separation of the plates by dx is thus

$$dW = F\,dx = \frac{1}{2}\epsilon_0 E^2\,A\,dx,$$

so that the stored energy in the capacitor at plate separation x is

$$W = \frac{1}{2}\epsilon_0 E^2\, Ax = \frac{1}{2}\epsilon_0 E^2 \times volume.$$

Thus

$$u_E = \frac{1}{2}\epsilon_0 E^2.$$

Problem 2:
Consider now a long straight solenoidal coil, with n turns of wire per unit length carrying a current I. The cross-sectional area of the solenoid is S. What is the inductance of a length l of the solenoid? Neglecting end-effects, what is the magnitude of the magnetic induction B in the region inside the solenoid? If the solenoid, which has a total inductance L is connected in series with a source $\mathcal{E}$ of emf and a resistor, the total ohmic resistance of the circuit being R, show that

$$\mathcal{E}I = I^2R + LI\frac{dI}{dt},$$

and hence show that if U_B is the energy stored in the magnetic field in the solenoid, $dU_B = LI\,dI$. Use this to derive $U_B = \frac{1}{2}LI^2$ and hence obtain

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0}$$

for the energy density in the magnetic field.
Answer:
Because the solenoid is long, if we neglect end-effects, we may conclude from the symmetry of the problem that the B-field is parallel to the axis. If one of the edges parallel to the axis is very far away from the solenoid, the line integral $\int \mathbf{B}\cdot d\mathbf{r}$ vanishes for the edge of the rectangle at great distance, and for the edges perpendicular to the axis of the solenoid. If the remaining edge of the rectangle is *outside* the solenoid, using Ampère's circuital law, $\oint \mathbf{B}\cdot d\mathbf{r} = \mu_0\times$ *(current flowing through path)*, which vanishes in this case, we see that B vanishes outside the solenoid. If on the other hand one of the edges parallel to the axis is *inside* the solenoid, the conclusion is that the B-field is uniform inside the solenoid, and of magnitude

$$B = \mu_0 nI.$$

This magnetic field links the turns of the solenoid, and if it changes (Faraday's law) there will be an induced *emf* given by $\mathcal{E}_i = -\frac{d}{dt}(N\phi)$, where N is the total number of turns of the solenoid, and ϕ is the flux through each of them, namely BS. Thus $\mathcal{E}_i = -\frac{d}{dt}(N\mu_0 nIS) = -\mu_0 nNS\frac{dI}{dt}$. The inductance of the solenoid is defined to be the coefficient of $-\frac{dI}{dt}$ in this equation, namely

$$L = \mu_0 nNS = \mu_0 n^2 S \times length\ of\ solenoid,$$

so that the inductance of length l is $\mu_0 n^2\, Sl$. The total *emf* driving the current I through the resistance R of the circuit described is $\mathcal{E} + \mathcal{E}_i$, so that Ohm's law gives (on multiplying by I) the desired equation

$$\mathcal{E}I = I^2R + LI\frac{dI}{dt}.$$

The left-hand side of this equation is just the rate at which the external *emf* $\mathcal{E}$ supplies energy to the circuit, and the first term on the right-hand side is the rate at which energy is dissipated in Ohmic heating. The remaining term must then be the rate at which energy is stored in the magnetic field inside the solenoid. Hence

$$\frac{dU_B}{dt} = LI\frac{dI}{dt}$$

or $dU_B = LI\,dI$ from which simple integration gives $U_B = \frac{1}{2}LI^2 = \frac{1}{2}\big(\mu_0 n^2 \times (volume\ of\ solenoid)\big)\times(\frac{B}{\mu_0 n})^2$. This energy is stored uniformly throughout the interior of the solenoid (neglecting end-effects), so that

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0}.$$