

LECTURE NOTES III

In this section of the notes we will discuss the simplest light waves (plane waves) in simple non-conducting media and review the concepts of polarisation and dispersion.

3.1 Plane Waves

Recall Maxwell's equations in the absence of sources -

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 0, \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}.\end{aligned}$$

Assume we have uniform isotropic linear media, for which  $\mathbf{D} = \epsilon \mathbf{E}$ ,  $\mathbf{B} = \mu \mathbf{H}$ , and write the electric and magnetic fields as

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i\mathbf{k}\mathbf{n}\cdot\mathbf{x} - i\omega t}, \quad \mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 e^{i\mathbf{k}\mathbf{n}\cdot\mathbf{x} - i\omega t},$$

where  $\mathbf{E}_0$  and  $\mathbf{B}_0$  are constant vectors and  $\mathbf{n}$  is a constant unit vector. Then Maxwell's equations are satisfied provided that (Exercise: prove this)

$$\begin{aligned}k^2 &= \mu\epsilon\omega^2, \\ \mathbf{n} \cdot \mathbf{E}_0 &= 0, \quad \mathbf{n} \cdot \mathbf{B}_0 = 0, \\ \mathbf{B}_0 &= \sqrt{\mu\epsilon}(\mathbf{n} \times \mathbf{E}_0)\end{aligned}$$

(we scale  $k$  so that  $\mathbf{n}$  is of unit length). These conditions mean that firstly, the electric and magnetic fields are perpendicular to the direction of propagation of the wave  $\mathbf{n}$ , ie this is a *transverse* wave. Furthermore, the electric and magnetic fields are perpendicular to each other (from the last equation). Also,  $c\mathbf{B}$  and  $\mathbf{E}$  have the same magnitude for these waves in the case of free space (for which  $\sqrt{\mu_0\epsilon_0} = 1/c$ ). This follows from taking the vector norm of the last equation, and using the above facts. It is straightforward to show that the energy flow per unit area per unit time is given by  $\frac{1}{\mu} \mathbf{E} \times \mathbf{B} = \sqrt{\epsilon/\mu} |\mathbf{E}_0 \cdot \mathbf{E}_0| \mathbf{n}$  and that the time-averaged energy density is  $\frac{1}{2}(\epsilon \mathbf{E}^2 + \frac{1}{\mu} \mathbf{B}^2) = \epsilon |\mathbf{E}_0 \cdot \mathbf{E}_0|$ , with the speed of energy flow thus being  $v = 1/\sqrt{\mu\epsilon}$ , the phase velocity. (The phase velocity is defined to be  $v = \frac{\omega}{k} = 1/\sqrt{\mu\epsilon} = c/n$ , and the index of refraction  $n$  is  $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$ .)

3.2 Polarisation

Using the results of the section above, we introduce a set of real mutually orthogonal unit vectors  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ , where  $\mathbf{e}_1$  is parallel to  $\mathbf{E}_0$  and  $\mathbf{e}_2$  is parallel to  $\mathbf{B}_0$ . Then we may write the solution discussed above as

$$\mathbf{E} = \mathbf{e}_1 E_0, \quad \mathbf{B} = \mathbf{e}_2 \sqrt{\mu\epsilon} E_0,$$

where  $E_0 = |\mathbf{E}_0|$ . We note that we may have alternatively written

$$\mathbf{E} = \mathbf{e}_2 E'_0, \quad \mathbf{B} = -\mathbf{e}_1 \sqrt{\mu\epsilon} E'_0,$$

as another solution of Maxwell's equations (this is just a  $90^\circ$  rotation of the first solution, with a different constant  $E'_0$ ). The first solution above is described as a *linearly polarised* wave, with polarisation vector  $\mathbf{e}_1$ , since the electric field is always in the  $\mathbf{e}_1$  direction. Note that, by definition, the plane or direction of polarisation of a wave equals the plane or direction of the electric field  $\mathbf{E}$ . Similarly, the second solution is linearly polarised with polarisation vector  $\mathbf{e}_2$ . We can combine the two plane waves to form the most general homogeneous plane wave propagating in the direction  $\mathbf{k} = k\mathbf{n}$ ,

$$\mathbf{E}(\mathbf{x}, t) = (\mathbf{e}_1 E_1 + \mathbf{e}_2 E_2) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t},$$

with the amplitudes  $E_1, E_2$  constants. Allowing these constants to be complex permits phase differences between waves of different polarisations. For example, consider the solutions

$$\mathbf{E}(\mathbf{x}, t) = E_1(\mathbf{e}_1 + i\mathbf{e}_2)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}.$$

The physical electric field (which is defined here to be the real part of the solution) is then constant in magnitude, but sweeps out a circle with frequency  $\omega$  as time evolves (at a given point in space). This is because  $\text{Re}\mathbf{E} = E_1\mathbf{e}_1 \cos(\omega t) + E_2\mathbf{e}_2 \sin(\omega t)$ . This is a *circularly polarised* wave (with positive helicity, since the rotation is clockwise for an observer facing the oncoming wave). In general there is a four parameter family of polarisations, described by the *Stokes parameters*. These describe the helicities (positive and negative) and the linear polarisations in the two directions perpendicular to the waves' motion.

### 3.3 Dispersion

*Dispersion* means that the effect of a medium on light traversing it depends on the frequency of that light. In the above we implicitly assumed that the parameters  $\mu, \epsilon$  were real and constants. This is thus the *dispersion free* case. As we saw, in this case, waves travel without distortion. In reality, all matter exhibits dispersion, with different frequencies of light being affected differently by the media.

As a simple model for the physics of dispersion, consider the model of matter as molecules localised at sites. An applied electric field produces an induced electric dipole moment in each molecule. Suppose that the electrons in each molecule are bound under the action of a restoring force  $\mathbf{F} = -m\omega_0^2\mathbf{x}$ , with  $m$  the mass of the charge and  $\omega_0$  the oscillation frequency. The equation of motion for the electron is

$$\frac{d^2\mathbf{x}}{dt^2} + \gamma\frac{d\mathbf{x}}{dt} + \omega_0^2\mathbf{x} = -\frac{e}{m}\mathbf{E}(\mathbf{x}, t)$$

( $\gamma$  is the damping force,  $-e$  the charge on the electron, the relative permeability has been taken to be one, and this is for small oscillations and magnetic forces). Assuming time dependence of  $e^{-i\omega t}$ , ie  $\mathbf{x} = \mathbf{x}_0 e^{-i\omega t}$ , we have the dipole moment for an electron as

$$\mathbf{p} = -e\mathbf{x}_0 = \frac{e^2}{m}(\omega_0^2 - \omega^2 - i\omega\gamma)^{-1}\mathbf{E}.$$

Hence the dielectric constant  $\epsilon = (1 + \chi_e)\epsilon_0$  is given by

$$\epsilon(\omega) = \epsilon_0 + \frac{e^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}.$$

(Recall that  $\mathbf{p} = \epsilon_0\chi_e\mathbf{E}$  defines  $\chi_e$ .) This formula is the basis for a good description of the atomic contribution to the dielectric constant. Note that this  $\epsilon$  is now dependent upon the frequency  $\omega$ .  $\omega_0$  is called the *resonant frequency*.

Typically,  $\gamma$  is small in comparison to the frequency  $\omega_0$ , so that the imaginary part of  $\epsilon$  is small. In the normal case there is more than one resonant frequency (eg due to more than one electron or molecule). Then, for frequencies below the smallest resonance,  $\epsilon/\epsilon_0$  is greater than one – if  $\omega\gamma$  is small, then if  $\omega < \omega_0$  we have

$$\frac{\epsilon}{\epsilon_0} \sim 1 + \frac{e^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2} > 1.$$

As one increases frequency past resonances more negative terms appear and  $\epsilon/\epsilon_0$  eventually becomes less than one. In the neighbourhood of a resonant frequency,  $\epsilon/\epsilon_0$  is dominated by a large, purely imaginary term – as  $\omega$  increases to  $\omega_0$ , we have

$$\frac{\epsilon}{\epsilon_0} \sim \frac{e^2}{m\epsilon_0} \frac{i}{\omega\gamma}.$$

This corresponds physically to dissipation of energy from the electromagnetic field into the medium, and is called *resonant absorption*. (If the imaginary part of  $\epsilon(\omega)$  is negative, the opposite occurs, and the medium

gives energy to the electromagnetic wave, as occurs with lasers, for example.) Changes in the real part of  $\epsilon(\omega)$  are *dispersion*, described as *normal* when it increases and *anomalous* when it decreases.

### 3.4 Some Results from Fourier Analysis

For use in the next section we here recall some results from Fourier analysis. Given a suitably well-behaved function  $D(\mathbf{x}, \omega)$ , its *Fourier transform* (in  $t$ )  $D(\mathbf{x}, t)$  is given by

$$D(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} D(\mathbf{x}, \omega) e^{-i\omega t} d\omega.$$

The functions  $e^{i\omega t}$  form an orthogonal basis, satisfying the condition

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-t')} d\omega = \delta(t-t'),$$

where  $\delta(t)$  is the Dirac delta function which satisfies

$$\int_{-\infty}^{\infty} \delta(t-t') f(t) dt = f(t')$$

for any function  $f(t)$ . Using these results, one can prove that the inverse Fourier transform is given by

$$D(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} D(\mathbf{x}, t) e^{i\omega t} dt.$$

The *convolution* theorem in Fourier analysis gives an expression for the Fourier transform  $H(t)$  of the product of two functions  $f(\omega)g(\omega)$ , in terms of the transforms  $F(t)$ ,  $G(t)$  of the functions  $f$  and  $g$ . This formula is

$$H(t) = \int_{-\infty}^{\infty} F(t') G(t-t') dt'.$$

### 3.5 The Kramers-Krönig Relations

The dependence of  $\epsilon_r$  and hence of the refractive index  $n$  on the frequency  $\omega$  means that light of different colours propagates with different speeds, and this leads to the *dispersion* of light (the separation of colours by a prism for example). What we have is

$$\mathbf{D}(\mathbf{x}, \omega) = \epsilon_0 \epsilon_r(\omega) \mathbf{E}(\mathbf{x}, \omega).$$

This implies that

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0 \mathbf{E}(\mathbf{x}, t) + \epsilon_0 \int_{-\infty}^{\infty} G(\tau) \mathbf{E}(\mathbf{x}, t-\tau) d\tau,$$

where we have introduced

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\epsilon_r(\omega) - 1] e^{-i\omega\tau} d\omega.$$

As a model for the frequency dependence of  $\epsilon_r$ , consider what happens if the medium is made of molecules with a single oscillator electron (as discussed above):

$$\epsilon_r = 1 + \frac{\omega_P^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

( $\omega_P^2 = e^2/m\epsilon_0$ ). Then we have

$$G(\tau) = \frac{\omega_P^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega} d\omega.$$

The integral may be evaluated by the usual method of completing a contour in the complex plane; the integrand has poles at  $\omega = \omega_{\pm} = -\frac{i\gamma}{2} \pm \omega_R$  where  $\omega_R^2 = \omega_0^2 - \gamma^2/4$ . These poles lie in the lower half of the complex  $\omega$ -plane. For  $\tau < 0$ , the factor  $e^{-i\omega\tau}$  tends to zero along a semi-circle at infinity in the upper half-plane, and since there are no singularities in the upper half-plane, taking as contour of integration the familiar one composed from the real axis and the arc at infinity in the upper half-plane, we conclude that for  $\tau < 0$ ,  $G(\tau) = 0$ . On the other hand for  $\tau > 0$  we must close the contour in the lower half-plane to get convergence on the arc at infinity, and the contour then encloses the poles of the integrand (in the negative sense, which introduces an extra minus sign), giving with use of the residue theorem

$$\begin{aligned} G(\tau) &= -\frac{\omega_P^2}{2\pi} \oint \frac{e^{-i\omega\tau}}{-(\omega - \omega_+) (\omega - \omega_-)} d\omega \\ &= -\frac{\omega_P^2}{2\pi} (2\pi i) \left[ \frac{e^{-i\omega_+\tau}}{-(\omega_+ - \omega_-)} + \frac{e^{-i\omega_-\tau}}{-(\omega_- - \omega_+)} \right] \\ &= -\frac{\omega_P^2}{2\pi} (2\pi i) e^{-i(-i\gamma\tau/2)} \left[ \frac{e^{-i\omega_R\tau}}{-2\omega_R} + \frac{e^{+i\omega_R\tau}}{2\omega_R} \right] \\ &= \omega_P^2 e^{-\gamma\tau/2} \frac{\sin \omega_R\tau}{\omega_R}, \end{aligned}$$

for  $\tau > 0$ . Putting these results together

$$G(\tau) = \omega_P^2 e^{-\gamma\tau/2} \frac{\sin \omega_R\tau}{\omega_R} \theta(\tau). \quad (*)$$

The details are not so important as the presence of the factor  $\theta(\tau)$  which is clearly still present even in a more sophisticated model in which there would be a sum over similar terms (many oscillators, each contributing a term as above with an *oscillator strength*), or indeed in the full quantum mechanical treatment (which in effect gives an explanation of the oscillator strengths).

There is a ‘non-locality’ in time in the relation between  $\mathbf{D}$  and  $\mathbf{E}$  by of order  $\gamma^{-1} \approx 10^{-8}$  sec. for a typical spectral line;  $\gamma$  is the natural line width. The factor  $\theta(\tau)$  is essentially independent of the model, and it ensures that the response of the dielectric is *causally* related to the  $\mathbf{E}$ -field,

$$\frac{1}{\epsilon_0} \mathbf{D}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}, t) + \int_0^\infty G(\tau) \mathbf{E}(\mathbf{x}, t - \tau) d\tau,$$

since the lower limit of the  $\tau$ -integration may be taken as 0, not  $-\infty$ . We have  $G(\tau) \sim e^{-\gamma\tau/2}$ , with  $\gamma$  acting like a damping force. For  $\tau > 1/\gamma$ ,  $G$  is damped, and for times up to  $\tau \sim 1/\gamma$ ,  $E(t - \tau)$  affects  $D(t)$ . This then means that

$$\epsilon_r(\omega) = 1 + \int_0^\infty G(\tau) e^{i\omega\tau} d\tau$$

(this is just the inverse Fourier transform). Because  $G(\tau)$  is *real*, it is easy to see that

$$\epsilon_r(-\omega) = [\epsilon_r(\omega^*)]^*.$$

We also see that  $\epsilon_r$  is an analytic function of  $\omega$  in the upper half-plane (from equation (\*) above). The analyticity of  $\epsilon_r(\omega)$  may be extended to  $\Im\omega \geq 0$ . As  $\epsilon_r - 1$  is analytic, the Cauchy theorem then allows us to write (for any  $z$  inside  $C$ )

$$\epsilon_r(z) = 1 + \frac{1}{2\pi i} \oint_C \frac{[\epsilon_r(\omega') - 1]}{\omega' - z} d\omega'$$

in which the contour of integration can be taken to be the familiar contour made up from the real axis and a semicircle at infinity in the upper half-plane. The arc at infinity makes no contribution to the integral, because  $\epsilon_r - 1 \rightarrow 0$  sufficiently fast, and one has, for any  $z$  with  $\Im z > 0$

$$\epsilon_r(z) = 1 + \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\epsilon_r(\omega') - 1}{\omega' - z} d\omega'.$$

We take  $z = \omega + i\epsilon$ ,  $\epsilon > 0$ , and deduce

$$\epsilon_r(\omega) = \lim_{\epsilon \rightarrow 0} \left[ 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\epsilon_r(\omega') - 1}{\omega' - \omega - i\epsilon} d\omega' \right].$$

Now we use

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0 - i\epsilon} dx = P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx + i\pi f(x_0),$$

where  $P \int$  denotes the Cauchy principal value integral (recall the definition of this as  $P \int_a^b f(x) dx = \lim_{\delta \rightarrow 0^+} (\int_a^{c-\delta} f(x) dx + \int_{c+\delta}^b f(x) dx)$ , where the limit is well-defined). Thus

$$\epsilon_r(\omega) = 1 + \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\epsilon_r(\omega') - 1}{\omega' - \omega} d\omega'$$

or separating real and imaginary parts

$$\begin{aligned} \Re \epsilon_r(\omega) &= 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Im \epsilon_r(\omega')}{\omega' - \omega} d\omega' \\ \Im \epsilon_r(\omega) &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Re \epsilon_r(\omega') - 1}{\omega' - \omega} d\omega'. \end{aligned}$$

If we use also the previous result on  $\epsilon_r^*$  which amounts to

$$\begin{aligned} \Re \epsilon_r(-\omega) &= \Re \epsilon_r(\omega) \\ \Im \epsilon_r(-\omega) &= -\Im \epsilon_r(\omega) \end{aligned}$$

there follows the *Kramers-Krönig relations*

$$\begin{aligned} \Re \epsilon_r(\omega) &= 1 + \frac{2}{\pi} P \int_0^{\infty} \frac{\omega' \Im \epsilon_r(\omega')}{\omega'^2 - \omega^2} d\omega' \\ \Im \epsilon_r(\omega) &= -\frac{2\omega}{\pi} P \int_0^{\infty} \frac{\Re \epsilon_r(\omega') - 1}{\omega'^2 - \omega^2} d\omega' \end{aligned}$$

which had an important influence on the development of quantum mechanics, and which in their generalised form continue to be of importance in studies of high energy physics. Because in the optical case they give relations between the real part of  $\epsilon_r$ , which determines *dispersion*, and the imaginary part which determines *absorption*, (c.f. section (3.3)) these relations and their generalisations are called *dispersion relations*; they are a consequence of *analyticity* (here of  $\epsilon_r$ ), which in turn is intimately related to the *causality* of the physical process studied.