MSci 4261 ELECTROMAGNETISM
LECTURE NOTES IV

In this part of the notes, we will discuss how Maxwell's equations can be reduced to second-order equations for the vector and scalar potentials. We will exhibit the solutions to these equations by defining the Green functions.

4.1 Scalar and Vector Potentials

Recall the Maxwell equations in a vacuum with sources -

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{1}{\epsilon_0}\rho, \\ \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}. \end{aligned} \tag{2}$$

We can solve the first two equations by introducing the *vector potential* $\mathbf{A}$ and the *scalar potential* Φ, and writing

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{E} &= -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}. \end{aligned} \tag{3}$$

However, note that there is a redundancy in this solution - $\mathbf{B}$ and $\mathbf{E}$ are unchanged if we make the transformations

$$\begin{aligned} \mathbf{A} &\to \mathbf{A} + \nabla \Lambda, \\ \Phi &\to \Phi - \frac{\partial \Lambda}{\partial t}, \end{aligned} \tag{4}$$

for any function Λ. This redundancy is called a *gauge freedom*, and the transformations (4) are called *gauge transformations* or a *gauge symmetry*. Whilst this may seem a little recondite, gauge symmetries turn out to be central to our understanding of modern fundamental interactions, in particular in the Standard Model. They are also crucial to our understanding of the *quantum* electromagnetic field.
The other two Maxwell equations (2) can be written, using (3), as

$$\begin{aligned} \boldsymbol{\nabla}^2 \Phi + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) &= -\frac{1}{\epsilon_0}\rho \\ \boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) &= -\mu_0 \mathbf{J}. \end{aligned} \tag{5}$$

One way to proceed is to fix the gauge freedom (4) by imposing what is called a *gauge fixing* condition. This effectively fixes Λ. An example is the *Lorentz gauge*

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0. \tag{6}$$

The general idea here is to solve (5) for the potentials $\mathbf{A}, \Phi$ (in terms of the sources $\rho, \mathbf{J}$), and then (3) gives the electromagnetic fields $\mathbf{E}, \mathbf{B}$. In the Lorentz gauge, the equations (5) become

$$\begin{aligned} \Box \Phi := \boldsymbol{\nabla}^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\frac{1}{\epsilon_0}\rho \\ \Box \mathbf{A} := \boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J} \end{aligned} \tag{7}$$

($\Box$ is the d'Alembertian).

Now note the following:

(1) Notice that the left-hand sides of the two equations in (7) are very similar - in fact the potentials are part of a *four-vector* A^μ as we will see in a later lecture;

(2) The operator $\Box$ is a wave operator, and the equations (7) are wave equations;

(3) There are other gauge fixing conditions - for example the *Coulomb gauge*, which is $\nabla \cdot \mathbf{A} = 0$. In this gauge we obtain

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho$$

so here the potential Φ describes the instantaneous Coulomb potential due to the charge density ρ, and

$$\Box \mathbf{A} = -\mu \mathbf{J}_t, \qquad (*)$$

where $\mathbf{J}_t$ is transverse ($\nabla \cdot \mathbf{J}_t = 0$). Physically then, in the Coulomb gauge the scalar potential determines the "near" Coulomb fields in terms of Φ and the vector potential $\mathbf{A}$ determines the (tranverse) *radiation* fields in terms of $\mathbf{J}_t$.

A proof of (*) follows: Firstly note (see lecture notes 2)

$$\nabla^2 \frac{1}{|\mathbf{x}' - \mathbf{x}|} = -4\pi\delta^3(\mathbf{x}' - \mathbf{x})$$

(where the derivatives in ∇ are with respect to $\mathbf{x}$), and the identity

$$\nabla^2 \mathbf{V} = \nabla\nabla \cdot \mathbf{V} - \nabla \times \nabla \times \mathbf{V}$$

for any vector field $\mathbf{V}$. Thus

$$\begin{aligned} \mathbf{J}(\mathbf{x}) = \int \delta^3(\mathbf{x}' - \mathbf{x})\mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' &= \nabla^2 \int \frac{-1}{4\pi} \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \\ &= -\frac{1}{4\pi}\nabla\nabla \cdot \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' + \frac{1}{4\pi}\nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' \\ &=: \mathbf{J}_l + \mathbf{J}_t \end{aligned}$$

with $\boldsymbol{\nabla} \times \mathbf{J}_l = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{J}_t = 0$, so that these fields are longitudinal and transverse respectively. Now $\boldsymbol{\nabla}^2 \Phi = -\frac{1}{\epsilon_0}\rho$ in the Coulomb gauge, so that

$$\Phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}'$$

whence

$$\frac{1}{c^2}\, \boldsymbol{\nabla}\dot{\Phi} = \frac{1}{4\pi\epsilon_0 c^2}\boldsymbol{\nabla} \int \frac{\dot{\rho}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' = -\frac{\mu_0}{4\pi}\boldsymbol{\nabla} \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' = \mu_0 \mathbf{J}_l$$

(as $\boldsymbol{\nabla} \cdot \mathbf{J} + \rho = 0$). Thus (see (5))

$$\Box \mathbf{A} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\boldsymbol{\nabla}\dot{\Phi} = -\mu_0 \mathbf{J}_t.$$

4.2 The Delta-Function

Our discussion of Green functions will rely on use of Dirac delta functions. The delta-function is a continuum generalisation of the finite-dimensional Kronecker delta. Recall that this is defined by

$$\delta_{nn'} = \begin{cases} 1 & n = n' \\ 0 & n \neq n' \end{cases}$$

(n is an integer) so that

$$\sum_n \delta_{nn'} f_n = f_{n'}.$$

The continuous analogue of this (in one dimension) is denoted $\delta(x - x')$. This has the properties

$$\delta(x - x') = 0 \qquad x \neq x';$$

$$\delta(x - x') = \delta(x' - x);$$

and

$$\int dx\, \delta(x - x')\, f(x) = f(x')$$

for "all" functions f (there are some weak restrictions which will not concern us here). Some consequences of these properties are

$$\int \delta(x - x')\ dx' = 1$$

if the integration range includes the value $x' = x$ (the integral is zero if this is not the case),

$$\delta(cx) = \frac{1}{c}\delta(x),$$

(c is a constant), and

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\Big(\delta(x - a) + \delta(x + a)\Big)$$

(a is a constant) which we will need in the following. The proof of this relation follows by noting that

$$f(a^2) = \int_{-\infty}^{\infty} \delta(y - a^2) f(y) dy = \int_0^{\infty} \delta(y - a^2) f(y) dy = \int_0^{\infty} \delta(x^2 - a^2) f(x^2) 2x\ dx.$$

One can then check that this is the same as $\frac{1}{2|a|}(\delta(x - a) + \delta(x + a))$ (inserted in the same integral) in each of the cases $a > 0, a < 0$.

An explicit representation of the delta function, which we will use often in the following, is

$$\delta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\lambda\omega}.$$

(Using the Fourier transform formulae above, one can check that this representation satisfies

$$\int_{-\infty}^{\infty} \delta(\lambda - \lambda') f(\lambda) d\lambda = f(\lambda')$$

for any function f.)

The one-dimensional delta function defined above can be immediately generalised to higher dimensions, in Cartesian coordinates. For example, in three dimensions, define

$$\delta^3(\mathbf{x} - \mathbf{x}') = \delta(x - x')\delta(y - y')\delta(z - z')$$

where $\mathbf{x} = (x, y, z), \mathbf{x}' = (x', y', z')$. In other coordinates a Jacobian arises - for example, in spherical polars

$$\delta^3(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2 \sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')$$

(recall that $dxdydz = r^2 \sin\theta dr d\theta d\phi$). The Jacobian term is needed so that the properties of the delta function continue to be satisfied.

4.3 Green functions

In this section we will see how to formally solve wave equations using inverse differential operators, or Green functions. First consider the equation

$$\Box\mathbf{A} = \boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}.$$

Integrate this with $\int_{-\infty}^{\infty} dt e^{i\omega t}$ to get

$$(\boldsymbol{\nabla}^2 + k^2)\mathbf{A}(\mathbf{x}, \omega) = -\mu_0\mathbf{J}(\mathbf{x}, \omega) \tag{8}$$

in terms of the Fourier transform fields. We define $k^2 = \omega^2/c^2$.

We wish to solve this equation for the electromagnetic potentials $\mathbf{A}$, given the source fields $\mathbf{J}$.

Suppose that there is a function $G_k(\mathbf{x}, \mathbf{x}')$ satisfying

$$(\boldsymbol{\nabla}^2 + k^2)\; G_k(\mathbf{x}, \mathbf{x}') = -4\pi\delta^3(\mathbf{x} - \mathbf{x}'). \tag{9}$$

Such a G_k is called a *Green function* for the operator $\boldsymbol{\nabla}^2 + k^2$. One can think of $\delta^3(\mathbf{x} - \mathbf{x}')$ as the functional unit 1 (just like the Kronecker delta is the (infinite-dimensional but discrete) unit matrix), so that G_k is proportional to the *inverse* of the operator $\boldsymbol{\nabla}^2 + k^2$.
Then we have the result that

$$\mathbf{A}(\mathbf{x}, \omega) = \frac{\mu_0}{4\pi}\int G_k(\mathbf{x}, \mathbf{x}')\mathbf{J}(\mathbf{x}', \omega)d^3x', \tag{10}$$

is the solution of (8). (This may be viewed as solving (8) functionally by writing $\mathbf{A}$ as the inverse of the differential operator, ie the Green function, times the right-hand side of (8).) The above result is easy to see:

$$\begin{aligned}(\boldsymbol{\nabla}^2 + k^2)\mathbf{A}(\mathbf{x}, \omega) &= \frac{\mu_0}{4\pi}\int \Big((\boldsymbol{\nabla}^2 + k^2)_x G_k(\mathbf{x}, \mathbf{x}')\Big)\mathbf{J}(\mathbf{x}', \omega)d^3x' \\ &= -\mu_0\int \delta^3(\mathbf{x} - \mathbf{x}')\mathbf{J}(\mathbf{x}', \omega)d^3x' = -\mu_0\mathbf{J}(\mathbf{x}, \omega)\end{aligned}$$

Thus, to solve (8) for $\mathbf{A}$, one need only solve the simpler equation (9) for G_k, then use (10) to find $\mathbf{A}$.

To solve (9), note that $\boldsymbol{\nabla}^2 + k^2$ is $\mathbf{x}$-translation invariant, and so the solution of (9) can only depend upon the difference

$$\mathbf{r} = \mathbf{x} - \mathbf{x}'.$$

The operator $\boldsymbol{\nabla}^2 + k^2$ is also rotation invariant ($\boldsymbol{\nabla}^2$ is a scalar under rotations) so in fact the solution can only depend upon

$$r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|.$$

Now use spherical polar coordinates (r, θ, ϕ) (for $\mathbf{x} - \mathbf{x}'$). Then by the above arguments, G_k is a function of r only. In spherical polars,

$$\boldsymbol{\nabla}^2 G_k(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\Big(rG_k(r)\Big) + \Big(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\Big) \text{ terms}$$

so that (9) becomes

$$\frac{1}{r}\frac{d^2}{dr^2}\big(rG_k\big) + k^2 G_k = -4\pi\delta^{(3)}(\mathbf{r}). \tag{11}$$

To solve this, consider the case when $r \neq 0$. Then the equation above can be written

$$\frac{d^2}{dr^2}\big(rG_k\big) + k^2(rG_k) = 0,$$

which has solution

$$rG_k = Ae^{ikr} + Be^{-ikr},$$

with A, B constants.

When $r \to 0$, the $\frac{1}{r}$ term on the left-hand side of (11) dominates so that the equation becomes

$$\frac{1}{r}\frac{d^2}{dr^2}\big(rG_k\big) = -4\pi\delta^{(3)}(\mathbf{r}) \qquad (12).$$

This is just what we would get from Poisson's equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0},$$

if we took $\Phi = G_k$ and $\rho = 4\pi\epsilon_0\delta^{(3)}(\mathbf{r})$. The latter corresponds to a point charge of magnitude $q = 4\pi\epsilon_0$, located at the origin. The solution to this Poisson equation is then

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{1}{r} = \frac{1}{r},$$

so that the solution to (12) above is

$$G_k = \frac{1}{r}.$$

Combining the information above we conclude that the solution of (9) is

$$G_k = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}, \qquad \text{with } A + B = 1,$$

the condition on A, B arising from the requirement that $G_k \to \frac{1}{r}$ as $r \to 0$. Physical considerations will dictate the choice of constant A (or B).

Let

$$G^{(\pm)} = \frac{e^{\pm ikr}}{r}.$$

Since

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega t}\mathbf{A}(\mathbf{x}, \omega),$$

the time-dependent $\mathbf{A}(\mathbf{x}, t)$ will have $e^{-i\omega t}$ factors. Thus we consider the functions

$$G_k^{(\pm)}\, e^{-i\omega t} = \frac{e^{-i(\omega t \mp kr)}}{r}.$$

The function G_k^+ is an *outgoing* (from the interaction point $\mathbf{r} = \mathbf{x} - \mathbf{x}' = 0$) wave, which will be the normal physical choice, so that we would take $B = 0$. G_k^- is correspondingly an incoming wave.

Having found

$$G_k(r) = G_k^+(r) = \frac{e^{ikr}}{r}$$

then from (3) above the potential is given by

$$\mathbf{A}(\mathbf{x},\omega) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}',\omega)\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\ d^3x'.$$

Thus

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega t}\frac{\mu_0}{4\pi}\int d^3x' \int_{-\infty}^{\infty} dt'\ \mathbf{J}(\mathbf{x}',t')e^{i\omega t'}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}.$$

The ω integration involves

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega(t-t')}\frac{e^{ikr}}{r} = \frac{1}{2\pi r}\int_{-\infty}^{\infty} d\omega\ e^{i\omega(r/c-\tau)} = \frac{1}{r}\delta(\tau - r/c),$$

where

$$\tau := t - t'$$

(and $k = \omega/c$), using the representation of the delta function given earlier. Thus

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int_{-\infty}^{\infty} dt'\ \mathbf{J}(\mathbf{x}',t')\ \frac{1}{r}\ \delta\Big(t' - (t - r/c)\Big)$$

or

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\ \mathbf{J}(\mathbf{x}',t-r/c)\ \frac{1}{r}. \tag{13}$$

These are called the *retarded potentials.* Physically, what is happening is that values of the currents $\mathbf{J}$ at the point $\mathbf{x}'$ at time $t - r/c$ affect or disturb the electromagnetic potentials $\mathbf{A}$ at the point $\mathbf{x}$ at time t, where the disturbance travels at the speed of light, so that $ct = |\mathbf{x}-\mathbf{x}'| = r$.

Similarly, choosing the solution G_k^- in the above arguments, one gets the *advanced* potentials, equation (5) with $t + r/c$ replacing $t - r/c$ in $\mathbf{J}$.

To summarise, we can say that if we define

$$G^+(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta(\tau - r/c)}{r} = \frac{1}{|\mathbf{x}-\mathbf{x}'|}\ \delta\Big(t - (t' + \frac{1}{c}|\mathbf{x}-\mathbf{x}'|)\Big)$$

then

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt'\ G^+(\mathbf{x},t;\mathbf{x}',t')\ \mathbf{J}(\mathbf{x}',t') \tag{14}$$

solves

$$\Box\mathbf{A} = \left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0\mathbf{J}.$$

In equation (14) above the “response”of $\mathbf{A}$ at $(\mathbf{x},t)$, to the source $\mathbf{J}$ at $(\mathbf{x}',t')$ is “propagated” by the Green function $G^+(\mathbf{x},t;\mathbf{x}',t')$.

A very similar analysis applies to solving

$$\Box\Phi = -\frac{1}{\epsilon_0}\rho$$

for the scalar potential Φ in the Lorentz gauge. Thus we have solved Maxwell's equations, in the presence of sources, using the vector and scalar potentials, and Green functions. In the following lectures we will see how to perform an expansion of this solution in order to identify the different physical contributions.