MSci 4261 ELECTROMAGNETISM

LECTURE NOTES V

In this section of the notes, we will use the previous results on solving the wave equation in order to derive results for dipole and multipole radiation.

5.1 The multipole expansion

Suppose that the sources have time dependence

$$\rho(\mathbf{x}, t) = \rho(\mathbf{x})e^{-i\omega t},$$

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{J}(\mathbf{x})e^{-i\omega t}.$$

Then, in Lorentz gauge, $\Box\mathbf{A} = -\mu_0\mathbf{J}$, which is solved by

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x})e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3x'\ \mathbf{J}(\mathbf{x}')\ e^{-i\omega(t-r/c)}\ \frac{1}{r}$$

with $r = |\mathbf{x} - \mathbf{x}'|$. Thus

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x'\mathbf{J}(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}\frac{1}{|\mathbf{x}-\mathbf{x}'|}.$$

In order to proceed with this, we would like to expand the exponential in various cases, depending upon the magnitude of r compared with the wavelength of the light $\lambda = 2\pi c/\omega$. Assume that the sources are confined to a region of dimension d and that the wavelength of the light satisfies $\lambda >> d$.

We will define the

Near zone: Here $d << r << \lambda$, ie we consider distances which are small compared with the wavelength of the light; the

Intermediate zone: with $d << r \sim \lambda$; and the

Far zone with $d << \lambda << r$.

In the near zone, $kr = \omega r/c = 2\pi r/\lambda << 1$, and so $e^{ik|\mathbf{x}-\mathbf{x}'}| \approx 1$, so that

$$\mathbf{A} \approx \frac{\mu_0}{4\pi}\int\ d^3x'\ \mathbf{J}(\mathbf{x}')\frac{1}{|\mathbf{x}'-\mathbf{x}|}.$$

The near field is *quasi-stationary*; it oscillates harmonically with a time dependence $e^{-i\omega t}$, but is otherwise just the same as for the stationary case.

In the far zone, $kr = 2\pi r/\lambda >> 1$. Let $\mathbf{x} = r\mathbf{n}$ (note that thus $r = |\mathbf{x}|$ here). Then

$$|\mathbf{x} - \mathbf{x}'| = [(\mathbf{x} - \mathbf{x}')^2]^{\frac{1}{2}} = [r^2 - 2r\mathbf{n}\cdot\mathbf{x}' + \mathbf{x}'^2]^{\frac{1}{2}} = r - \mathbf{n}\cdot\mathbf{x}' + \ldots,$$

where the omitted terms are proportional to higher powers of $|\mathbf{x}'|/r \leq d/r$. Keeping only the first term, we thus find that

$$\mathbf{A} \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\ d^3x'\,\mathbf{J}(\mathbf{x}') = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\ d^3x'\,\mathbf{x}'\Big(\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{x}')\Big) = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\ d^3x'\,\mathbf{x}'(i\omega)\rho(\mathbf{x}')$$
$$= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,i\omega\,\mathbf{p},$$

where we have integrated by parts in the first step, using the fact that the sources vanish outside a localised region, and in the second step we have used the conservation of charge equation $\boldsymbol{\nabla}\cdot\mathbf{J} = -\dot{\rho}$, and assumed that all fields have time dependence $e^{-i\omega t}$. In the above, $\mathbf{p}e^{-i\omega t} \equiv \int\ d^3x'\,\mathbf{x}'\,\rho(\mathbf{x}')e^{-i\omega t}$ is the *electric dipole* *moment* of the source. The dominant contribution to the fields in the far zone for a small source thus comes from the (oscillating) electric dipole moment. (The size of the dipole is of order d, which is small compared to r, but it is the radiation from the dipole, reaching the point $\mathbf{x}$, which is being realised by this contribution to $\mathbf{A}$.)

We comment that a similar calculation can be followed for the *scalar* potential Φ. This satisfies, in the Lorentz gauge,

$$\Box\Phi = -\frac{1}{\epsilon_0}\rho.$$

Using analogous arguments to those above, with the retarded solution, we find

$$\Phi = \frac{1}{4\pi\epsilon_0}\int d^3x' \int dt' \frac{\rho(\mathbf{x}')e^{-i\omega t'}}{|\mathbf{x}'-\mathbf{x}|}\delta\Big(t' + \frac{|\mathbf{x}'-\mathbf{x}|}{c} - t\Big).$$

In the far zone, an expansion in powers of $|\mathbf{x}'|/r$ is again possible. The zeroth order term obtained by replacing $|\mathbf{x}'-\mathbf{x}|$ by r is

$$\Phi = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\mathbf{x}')e^{-i\omega t'}}{r},$$

with $t' = t - r/c$. This is just the *Coulomb potential* produced by a charge $Q = \int d^3x' \, \rho(\mathbf{x}', t')$ placed at the origin, and this total charge is constant, ie

$$\Phi = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

and there is no resultant radiation to the lowest order (a dipole term enters in the next order of approximation however). Thus the dominant contribution to the scalar potential in the far zone is just given by the Coulomb potential coming from the total charge of the sources.

We have seen in the above that one may formally solve Maxwell's equations in the presence of sources by introducing potentials and using Green functions to solve the equations for the potentials. The final stage is to derive the values for the electromagnetic fields $\mathbf{B}, \mathbf{E}$ themselves, using the potentials. The expression for $\mathbf{B}$ is simple:

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}.$$

We can sidestep the use of Φ in finding $\mathbf{E}$ by noting that $\mathbf{J} = 0$ outside the sources, so that one has $\dot{\mathbf{E}} = c^2\boldsymbol{\nabla}\times\mathbf{B}$. Then, using the $e^{-i\omega t}$ time dependence of the fields one has

$$\mathbf{E} = \frac{ic}{k}\boldsymbol{\nabla}\times\mathbf{B}.$$

Hence, to find $\mathbf{B}$ and $\mathbf{E}$ from $\mathbf{A}$ we need only apply the curl operator twice. In the following we will see the results for some basic cases.

5.2 Electric Dipole Radiation

As we saw in the previous lectures, the electric dipole contribution is the lowest order contribution in a systematic expansion - the *multipole expansion* - of the radiation from a general source.

For the case of pure electric dipole radiation, we found

$$\mathbf{A} = -\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\frac{ik}{c}\mathbf{p}.$$

This results in the magnetic field

$$\begin{aligned}\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} &= -\frac{1}{4\pi\epsilon_0}\frac{ik}{c}\Big[\boldsymbol{\nabla}\Big(\frac{e^{ikr}}{r}\Big)\Big]\times\mathbf{p} \\ &= \frac{1}{4\pi\epsilon_0}\frac{k^2}{c}\frac{e^{ikr}}{r}\Big(1 - \frac{1}{ikr}\Big)\mathbf{n}\times\mathbf{p}.\end{aligned}$$

For the electric field we find

$$\begin{aligned}
\mathbf{E} = \frac{ic}{k}\boldsymbol{\nabla}\times\mathbf{B} &= \frac{ic}{k}\boldsymbol{\nabla}\times\Big[\frac{1}{4\pi\epsilon_0}\frac{k^2}{c}\frac{e^{ikr}}{r}\Big(1-\frac{1}{ikr}\Big)\mathbf{n}\times\mathbf{p}\Big] \\
&= \frac{ic}{k}\frac{1}{4\pi\epsilon_0}\frac{k^2}{c}\Big\{\frac{e^{ikr}}{r}\Big(1-\frac{1}{ikr}\Big)\boldsymbol{\nabla}\times(\mathbf{n}\times\mathbf{p}) - (\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{\partial}{\partial r}\Big[\frac{e^{ikr}}{r}\Big(1-\frac{1}{ikr}\Big)\Big]\Big\} \\
&= \frac{ik}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\Big\{-\frac{1}{r}[\mathbf{p}+(\mathbf{p}\cdot\mathbf{n})\mathbf{n}]\Big(1-\frac{1}{ikr}\Big) + [(\mathbf{n}\cdot\mathbf{p})\mathbf{n}-\mathbf{p}]\Big(ik-\frac{2}{r}+\frac{2}{ikr^2}\Big)\Big\} \\
&= \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\Big(\frac{1}{r^2}-\frac{ik}{r}\Big)[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}].
\end{aligned}$$

In the above, we have used $\boldsymbol{\nabla}\times(f(r)\mathbf{n}\times\mathbf{p}) = f(r)\boldsymbol{\nabla}\times(\mathbf{n}\times\mathbf{p}) + (\boldsymbol{\nabla}f)\times(\mathbf{n}\times\mathbf{p})$, for any function $f(r)$, and $\boldsymbol{\nabla}\cdot\mathbf{n} = 2/r$ and $\boldsymbol{\nabla}\times(\mathbf{n}\times\mathbf{p}) = -\frac{1}{r}\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})\mathbf{n}$. The results above for the electric and magnetic fields are quite complex, but these are exact for a pure electric dipole, and they simplify greatly in the near and far zone approximations. (Note that the fields below also have time dependence $e^{-i\omega t}$, which is suppressed for simplicity.) In the near zone $kr << 1$ we find

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{ik}{c}\mathbf{n}\times\mathbf{p}\frac{1}{r^2}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\Big[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}\Big]\frac{1}{r^3},$$

which are the quasistatic fields from the electric dipole and the concomitant current.

In the far zone $kr >> 1$ we have

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{k^2}{c}\mathbf{n}\times\mathbf{p}\frac{e^{ikr}}{r}$$

$$\mathbf{E} = c\mathbf{B}\times\mathbf{n}.$$

Notice that this is electromagnetic radiation - the fields $\mathbf{E}$, $\mathbf{B}$, and the direction of motion of the wave $\mathbf{n}$, are mutually perpendicular. The fields also fall off with distance like $\frac{1}{r}$.

Now consider the energy radiated by an electric dipole. The energy flux is given by the Poynting vector as usual. We take the far zone fields, as we will be interested in the total energy radiated, which may be found by considering the fields crossing a sphere at large distance.

Putting back the (real) time-dependence, we have for the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}\Big[\Re(\mathbf{E}e^{-i\omega t})\times\Re(\mathbf{B}e^{-i\omega t})\Big].$$

We will *time average* this. The average value of $\cos^2\theta$ over one period is 1/2 - since

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{1}{2}$$

(or one may just note that the averages of $\cos^2\theta$ and $\sin^2\theta$ are equal, and add to 1, so each must be equal to 1/2). Hence, for the time-averaged energy flux,

$$\langle\mathbf{S}\rangle = \frac{1}{2}\frac{1}{\mu_0}\Re\big[\mathbf{E}\times\mathbf{B}^*\big] = \frac{1}{2}\frac{1}{\mu_0 c}|\mathbf{E}|^2\mathbf{n}.$$

Now, if $\mathbf{n}$ is the unit normal to the sphere at radius r, then the area element is

$$d\boldsymbol{\Omega} = r^2\mathbf{n}d\Omega,$$

where $d\Omega$ is the solid angle (the r^2 factor is there since the total area of the sphere's surface is $4\pi r^2$, and the total solid angle is 4π). Then the power radiated through the area element $\mathbf{\Omega}$ is

$$\begin{aligned}\frac{dP}{d\Omega} &= r^2\mathbf{n}\cdot\langle\mathbf{S}\rangle = \frac{1}{2\mu_0 c}|r\mathbf{E}|^2 \\ &= \frac{1}{2\mu_0 c}\Big(\frac{1}{4\pi\epsilon_0}k^2\mathbf{n}\times\mathbf{p}\Big)^2 = \frac{1}{2}\frac{\mu_0}{16\pi^2 c}|\mathbf{p}|^2\omega^4\sin^2\theta,\end{aligned}$$

where θ is the angle between the orientation of the dipole (the direction of $\mathbf{p}$) and the direction where one is measuring the radiation (given by $\mathbf{n}$). Note the characteristic dipole angular $\sin^2\theta$ dependence - if one maps the lines of constant power one sees a dual-lobe structure with maxima perpendicular to the dipole (ie at $\theta = 0, \pi$). The frequency dependence is ω^4 ($\omega = kc = 2\pi c/\lambda = 2\pi\nu$), which is also characteristic.

The total power radiated is given by integrating the above expression over all directions (θ, ϕ), using $d\Omega = \sin\theta d\theta d\phi$. This results in

$$P = \frac{\mu_0}{12\pi c}\mathbf{p}^2\omega^4.$$

5.3 Magnetic dipole radiation

One can consider further terms in the expansion beyond the electric dipole term, and must do so if the dipole moment of the sources is zero (ie if $\mathbf{p} = 0$). Recall that

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x'\,\mathbf{J}(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}\frac{1}{|\mathbf{x}-\mathbf{x}'|}.$$

In the far zone, $kr >> 1 >> kd$, where the sources are localised in a region of dimensions d. Then

$$|\mathbf{x}-\mathbf{x}'| = r - \mathbf{n}\cdot\mathbf{x}' + ...$$

where the dots indicate terms of higher powers in $|\mathbf{x}'|/r$ and r here is $|\mathbf{x}|$. Using this one finds that

$$e^{ik|\mathbf{x}-\mathbf{x}'|}\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{e^{ikr}}{r}\Big(1 - ik\mathbf{n}\cdot\mathbf{x}' + ...\Big).$$

The first term (the "1" inside the brackets above) led to the electric dipole term which we studied earlier. The term of next highest order in the expansion of $|\mathbf{x}-\mathbf{x}'|^{-1}$ then leads to

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int d^3x'\,\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}').$$

Note the identity

$$\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') = \frac{1}{2}\Big[\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') + \big(\mathbf{n}\cdot\mathbf{J}(\mathbf{x}')\big)\mathbf{x}'\Big] + \frac{1}{2}\Big[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')\Big]\times\mathbf{n} \qquad (*)$$

which follows from expanding the double cross product term. This enables us to separate two contributions at this order - the magnetic dipole and the electric quadrupole. The second term on the right-hand side of the above expression gives rise to

$$\mathbf{A}^{\text{m.d.}} = ik\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{n}\times\mathbf{m},$$

where

$$\mathbf{m} \equiv \frac{1}{2}\int d^3x'\,[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')].$$

This is the *magnetic dipole moment* of the source.

To find the electric and magnetic radiation fields for a magnetic dipole source, one would then calculate

$$\mathbf{B}^{m.d.} = \boldsymbol{\nabla} \times \mathbf{A}^{m.d.},$$

$$\mathbf{E}^{m.d.} = \frac{ic}{k} \boldsymbol{\nabla} \times \mathbf{B}^{m.d.}.$$

However, we have already done these calculations, as one sees by the following - first notice that

$$\mathbf{A}^{m.d.}(\mathbf{m}) = \frac{i}{kc} \mathbf{B}^{e.d.}(\mathbf{p} \to \mathbf{m}),$$

ie the potential $\mathbf{A}^{m.d.}(\mathbf{m})$ for a magnetic dipole $\mathbf{m}$ is proportional to expression for the magnetic field of an electric dipole, with the electric dipole moment $\mathbf{p}$ replaced by the magnetic dipole moment $\mathbf{m}$. Then

$$\mathbf{B}^{m.d.} = \boldsymbol{\nabla} \times \mathbf{A}^{m.d.} = \frac{i}{kc} \boldsymbol{\nabla} \times \mathbf{B}^{e.d.}(\mathbf{p} \to \mathbf{m}) = \frac{i}{kc} \frac{k}{ic} \mathbf{E}^{e.d.}(\mathbf{p} \to \mathbf{m}) = \frac{1}{c^2} \mathbf{E}^{e.d.}(\mathbf{p} \to \mathbf{m}).$$

Also,

$$\begin{aligned}
\mathbf{E}^{m.d.} &= \frac{ic}{k} \boldsymbol{\nabla} \times \mathbf{B}^{m.d.} = \frac{ic}{k} \frac{1}{c^2} \boldsymbol{\nabla} \times \mathbf{E}^{e.d.}(\mathbf{p} \to \mathbf{m}) \\
&= -\frac{1}{k^2} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}^{e.d.}(\mathbf{p} \to \mathbf{m})) = -\frac{1}{k^2} \Big(-\boldsymbol{\nabla}^2 \mathbf{B}^{e.d.}(\mathbf{p} \to \mathbf{m}) + \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \mathbf{B}^{e.d.}(\mathbf{p} \to \mathbf{m}) \Big) \\
&= \frac{1}{k^2} \boldsymbol{\nabla} \times \Big(\boldsymbol{\nabla}^2 \mathbf{A}^{e.d.}(\mathbf{p} \to \mathbf{m}) \Big) = \frac{1}{k^2} \Big(-\frac{\omega^2}{c^2} \boldsymbol{\nabla} \times \mathbf{A}^{e.d.}(\mathbf{p} \to \mathbf{m}) \Big) \\
&= -\mathbf{B}^{e.d.}(\mathbf{p} \to \mathbf{m}).
\end{aligned}$$

Hence we conclude that

$$\begin{aligned}
\mathbf{B}^{m.d.} &= \frac{1}{c^2} \mathbf{E}^{e.d.}(\mathbf{p} \to \mathbf{m}), \\
\mathbf{E}^{m.d.} &= -\mathbf{B}^{e.d.}(\mathbf{p} \to \mathbf{m}).
\end{aligned}$$

This exhibits the similarity between electric and magnetic dipole radiation fields. For each, the fields $\mathbf{E}$ and $\mathbf{B}$ and the direction of radiation $\mathbf{n}$ are mutually perpendicular, and the angular and frequency dependence of the radiations are the same. The difference is in the polarisations (directions of the electric fields) - for electric dipole radiation the polarisation is in the same plane as that defined by the electric dipole and the vector $\mathbf{n}$, whereas for magnetic dipole radiation, the polarisation is perpendicular to the plane defined by the magnetic dipole and $\mathbf{n}$.

5.4 Electric quadrupole radiation

There is another contribution at the same order, arising from the first term on the right-hand side of the expression in (*) above. This gives

$$\begin{aligned}
\mathbf{A}^{e.q.} &= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \frac{1}{2} \int d^3x' \left[\mathbf{J}(\mathbf{x}')(\mathbf{n} \cdot \mathbf{x}') + (\mathbf{n} \cdot \mathbf{J}) \mathbf{x}' \right] \\
&= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \frac{-k^2 c}{2} \int d^3x' \, \mathbf{x}'(\mathbf{n} \cdot \mathbf{x}') \rho(\mathbf{x}'),
\end{aligned}$$

where there has been an integration by parts, and use made of the continuity equation $\boldsymbol{\nabla} \cdot \mathbf{J} = -\dot{\rho} = i\omega\rho$ for the time dependence under consideration. The above expression is proportional to the second moment of the charge distribution, and so may be identified with the contribution of the *electric quadrupole*.

5.5 Radiation from an Antenna

We will consider as a further example the radiation field from a centre-fed linear antenna. Assuming that the antenna is thin, and that radiation losses are small, the current density in the antenna may be represented by

$$\mathbf{j}(\mathbf{x},t) = I\,\sin\Big(\frac{kd}{2} - k|z|\Big)\delta(x)\delta(y)\hat{\mathbf{z}}e^{-i\omega t}.$$

(d is the linear dimension of the antenna, which is oriented in the z direction.) Then from our previous results, the radiation in the far zone is obtained from

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x'\,\mathbf{j}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

with

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{e^{ikr}}{r}e^{-ik\mathbf{n}\cdot\mathbf{x}'}.$$

Using $\mathbf{n}\cdot\mathbf{x}' = z'\cos\theta$ and introducing the expression above for the current density leads to

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}I\frac{e^{ikr}}{r}\int_{-d/2}^{d/2} dz'\,\sin\big(\frac{kd}{2} - k|z'|\big)e^{-ikz'\cos\theta} \\ &= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}I\frac{e^{ikr}}{r}\frac{2}{k}\Big[\frac{\cos\big(\frac{kd}{2}\cos\theta\big) - \cos\big(\frac{kd}{2}\big)}{\sin^2\theta}\Big].\end{aligned}$$

Using $\mathbf{B} \approx ik\mathbf{n}\times\mathbf{A} \Rightarrow |\mathbf{B}| = k\sin\theta|\mathbf{A}| = |\mathbf{E}|/c$, we find for the time-averaged power radiated

$$\begin{aligned}\Big(\frac{dP}{d\Omega}\Big) &= \frac{1}{2}\frac{1}{\mu_0 c}|r\mathbf{E}|^2 = \frac{1}{2}\frac{1}{\mu_0 c}|kcr\sin\theta\mathbf{A}|^2 \\ &= \frac{\mu_0 c}{8\pi^2}I^2\Big[\frac{\cos\big(\frac{kd}{2}\cos\theta\big) - \cos\big(\frac{kd}{2}\big)}{\sin\theta}\Big]^2.\end{aligned}$$

For a *half-wave antenna*, with $kd = \pi$,

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2}I^2\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta},$$

whereas for a *full-wave antenna*, with $kd = 2\pi$, the result is

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2}I^2\frac{4\cos^4(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}.$$

The angular distributions are thereby changed; more subtle changes in the radiation pattern can be achieved, e.g., by phased arrays of antennae.