MSci 4261 ELECTROMAGNETISM

LECTURE NOTES IX

Here we discuss the electromagnetic fields produced by the motion of charged particles.

9.1 Fields From a Static Source If the source charge-current density j^μ is independent of time, i.e., it is static, then the resulting fields will also be time-independent:

$$A^\mu(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x'\, j^\mu(\mathbf{x}') \frac{1}{r} = A^\mu(\mathbf{x}),$$

and

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= -\dot{\mathbf{A}} - \boldsymbol{\nabla}\Phi \\ &= -\boldsymbol{\nabla}(cA^0) \\ &= -\frac{1}{4\pi\epsilon_0} \boldsymbol{\nabla} \int d^3x'\, \rho(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x', \end{aligned}$$

which is after all just Coulomb's law; and

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \boldsymbol{\nabla} \times \mathbf{A} \\ &= \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \times \int d^3x'\, \mathbf{j}(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|} \\ &= \frac{\mu_0}{4\pi} \int \mathbf{j}(\mathbf{x}') \times \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x', \end{aligned}$$

which is just the Ampère-Biot-Savart law.

9.2 Fields Produced by a Moving Charged Particle

Suppose that the source of the fields is a point particle carrying a charge q which follows the trajectory $\mathbf{r} = \mathbf{r}(t)$. Then the source densities are

$$\begin{aligned} j^0 &= cq\, \delta^{(3)}[\mathbf{x} - \mathbf{r}(t)] \\ \mathbf{j} &= q\mathbf{u}(t)\, \delta^{(3)}[\mathbf{x} - \mathbf{r}(t)], \end{aligned}$$

where $\mathbf{u} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$. If we recall that the particle's 4-velocity U^μ is given by

$$U^\mu = (U^0 = \gamma_u c,\ \mathbf{U} = \gamma_u \mathbf{u}) = \gamma_u(c,\ \mathbf{u})$$

these may be combined to give

$$j^\mu(\mathbf{x}, t) = q\frac{1}{\gamma_u} U^\mu \delta^{(3)}[\mathbf{x} - \mathbf{r}(t)].$$

To make this manifestly covariant, we introduce the *proper time* τ along the particle's trajectory, i.e., the time which would be measured out by a clock moving with the particle. As the particle moves, its position (in the original frame, K say) is given by $\mathbf{r}(t)$, but also the time-interval $d\tau$ is related to dx^0 by

$$\frac{d\tau}{dx^0} = \frac{1}{c\gamma_u},$$

so that we may write

$$j^\mu(x) = cq \int d\tau\, U^\mu(\tau) \delta^{(4)}[x - r(\tau)],$$

where x is the 4-vector with components x^α, and $r^\alpha(\tau) = [ct, \mathbf{r}(t)]$, $\quad U^\alpha = [\gamma_u c, \gamma_u \mathbf{u}]$. This follows from

$$c \int d\tau\, U^\mu(\tau)\delta[x^0 - r^0] = c\int \frac{d\tau}{dx^0}\, dx^0 U^\mu \delta(x^0 - ct) = \frac{1}{\gamma_u} U^\mu.$$

For the fields we then have (applying out-going boundary conditions):

$$\begin{aligned} A^\mu(x) &= \mu_0 \int d^4x'\, D_r(x - x')\, cq \int d\tau\, U^\mu(\tau)\delta^{(4)}[x' - r(\tau)] \\ &= c\mu_0 q \int d\tau\, U^\mu(\tau) D_r[x - r(\tau)] \\ &= \frac{c\mu_0 q}{2\pi} \int d\tau\, U^\mu(\tau)\delta[(x - r(\tau))^2]\theta[x^0 - r^0(\tau)]. \end{aligned}$$

The delta-function then ensures that the field at x is determined by what happens on the trajectory of the particle only where $r(\tau)$ is light-like separated from x. This means that the point $r(\tau)$ lies on the light- cone with its vertex at x. There are *two* such points, one in the past of x, one to the future: the theta-function then picks out the *unique* point at which the trajectory crosses the *past* light-cone with vertex at x. When we come to evaluate $\int d\tau\, U(\tau)\delta[f(\tau)]$ there would in general be a sum over contributions from each of the roots of $f = 0$, but in the present case there is thus only the one contributing root, and we have

$$\int d\tau\, U(\tau)\delta[f(\tau)] = \int \frac{d\tau}{df} U(\tau)\delta(f)\, df = \Big[\frac{U(\tau)}{|df/d\tau|}\Big]\Big|_{f=0}.$$

Since in the present case $f = [x - r(\tau)]^2$, we have

$$\begin{aligned} \left|\frac{df}{d\tau}\right| &= | - 2[x - r(\tau)]_\beta \frac{dr^\beta}{d\tau}| \\ &= |2[x - r(\tau)]_\beta U^\beta(\tau)| \end{aligned},$$

and thus

$$A^\mu(x) = \frac{c\mu_0 q}{4\pi}\Big[\frac{U^\mu(\tau)}{U(\tau)\cdot[x - r(\tau)]}\Big]_{\tau=\tau_0},$$

where τ_0 is defined by

$$[x - r(\tau_0)]^2 = 0 \quad \text{and} \quad x^0 > r^0(\tau_0).$$

This is then manifestly covariant.

9.3 The Liénard-Wiechert Potentials

We can render this last expression more useful as follows. The light-cone condition means

$$[x - r(\tau_0)]^2 = [x^0 - r^0(\tau_0)]^2 - [\mathbf{x} - \mathbf{r}(\tau_0)]^2 = 0,$$

or in terms of $R \equiv |\mathbf{x} - \mathbf{r}(\tau_0)|$,

$$x^0 - r^0(\tau_0) = R,$$

where we have used the condition $x^0 > r^0(\tau_0)$ to fix the sign. Note that this is a complicated equation for τ_0 which determines the (unique) point to the past of the field point x on the trajectory of the particle from which an influence propagating at the speed of light reaches the position $\mathbf{x}$ at the time $ct = x^0$. The corresponding time r^0/c on the trajectory is called the *retarded time*, t_{ret}. Thus

$$c(t - t_{\text{ret}}) = R.$$

We also have

$$
\begin{aligned}
U \cdot [x - r(\tau_0)] &= U^0[x^0 - r^0(\tau_0)] - \mathbf{U} \cdot [\mathbf{x} - \mathbf{r}(\tau_0)] \\
&= U^0 R - \mathbf{U} \cdot [\mathbf{x} - \mathbf{r}(\tau_0)] \\
&= \gamma_u c R(1 - \boldsymbol{\beta} \cdot \mathbf{n}),
\end{aligned}
$$

where we have as usual written $\mathbf{u} = c\boldsymbol{\beta}$, and have defined the unit vector $\mathbf{n}$ which points from the point on the trajectory $\mathbf{r}(\tau_0)$ to the field point $\mathbf{x}$ by

$$
\mathbf{x} - \mathbf{r}(\tau_0) = \mathbf{n}R.
$$

This results in

$$
A^\mu = \frac{c\mu_0 q}{4\pi} \left\{ \frac{[1, \boldsymbol{\beta}]}{R(1 - \boldsymbol{\beta} \cdot \mathbf{n})} \right\}_{\tau=\tau_0}.
$$

In terms of the scalar Φ and vector $\mathbf{A}$, this is usually written as

$$
\Phi = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R} \frac{1}{1 - \boldsymbol{\beta} \cdot \mathbf{n}} \right]_{\text{ret}},
$$

$$
\mathbf{A} = \frac{\mu_0 q}{4\pi} \left[\frac{\mathbf{u}}{R} \frac{1}{1 - \boldsymbol{\beta} \cdot \mathbf{n}} \right]_{\text{ret}},
$$

where the suffix 'ret' indicates that the time variable to be used is the retarded time $t_{\text{ret}} = t - R/c$. These potentials are known as the *Liénard-Wiechert potentials.*

The Liénard-Wiechert potentials (or a direct calculation) may then be used to obtain

$$
\mathbf{B} = \frac{1}{c} [\mathbf{n} \times \mathbf{E}]_{\text{ret}},
$$

$$
\begin{aligned}
\mathbf{E} =& \frac{q}{4\pi\epsilon_0} \left[\frac{\mathbf{n} - \boldsymbol{\beta}}{\gamma_u^2 R^2 (1 - \boldsymbol{\beta} \cdot \mathbf{n})^3} \right]_{\text{ret}} \\
&+ \frac{q}{4\pi\epsilon_0} \frac{1}{c} \left[\frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^3 R} \right]_{\text{ret}}
\end{aligned}
$$

in which

$$
c\dot{\boldsymbol{\beta}} = \frac{d}{dt} \mathbf{u}
$$

is the acceleration of the particle. The contribution proportional to this is called the *acceleration field*, the other term the *velocity field.* Note also that the velocity field falls away with increasing separation R like R^{-2}, whilst the acceleration field falls away more slowly, like R^{-1}; for this reason the acceleration field is dominant in what is called the far field region.

9.4 Fields Produced by a Moving Charged Particle: The Oscillator

An important special case is that of the radiation from a charge executing simple harmonic motion — the charged *oscillator*. Let us write

$$
\mathbf{r}(t) = \mathbf{a}\, e^{-i\omega t},
$$

where we use the usual convention that the real part is to be taken, so that this really means $\mathbf{r} = \mathbf{a} \cos \omega t$. Then

$$
\boldsymbol{\beta} = \frac{-\mathbf{a}}{c} i\omega\, e^{-i\omega t},
$$

$$
\dot{\boldsymbol{\beta}} = -\frac{\mathbf{a}}{c} \omega^2\, e^{-i\omega t}.
$$

Let us also suppose $|\mathbf{a}|\omega << c$, so that the oscillator's motion is never relativistic. Then at all times $1 - \boldsymbol{\beta}\cdot\mathbf{n} \approx 1$. Also if we consider the field far from the oscillator, where $|\mathbf{x}| >> |\mathbf{a}|$, the distance R is always well approximated by $|\mathbf{x}|$. We then find

$$\begin{aligned}
\mathbf{B} =& \frac{1}{c}\mathbf{n}\times\mathbf{E} \\
\mathbf{E} \approx& \frac{q}{4\pi\epsilon_0}\frac{\mathbf{n}}{R^2} \\
&+ \frac{q}{4\pi\epsilon_0}\frac{1}{c}\mathbf{n}\times(\mathbf{n}\times\mathbf{a})\Big(-\frac{\omega^2}{c}\Big)\frac{e^{-i\omega t}}{R}.
\end{aligned}$$

In these equations $\mathbf{x} = R\mathbf{n}$. The term proportional to $\frac{1}{R^2}$ is called the *near field*, and that proportional to $\frac{1}{R}$ is the *far*, or *radiation field*. The *Poynting vector* is

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \\
&= \frac{1}{\mu_0}\Big[\frac{q}{4\pi\epsilon_0}\Big(-\frac{\omega^2}{c^2}\Big)\frac{\cos\omega t}{R}\Big]^2\frac{1}{c}[\mathbf{n}\times(\mathbf{n}\times\mathbf{a})]\times[\mathbf{n}\times\big(\mathbf{n}\times(\mathbf{n}\times\mathbf{a})\big)] \\
&= \frac{\mu_0 q^2\omega^4\cos^2\omega t}{16\pi^2 cR^2}[a^2 - (\mathbf{n}\cdot\mathbf{a})^2]\mathbf{n} \\
&= \frac{\mu_0 q^2}{16\pi^2 c}\frac{a^2}{R^2}\omega^4\cos^2\omega t\sin^2\theta\,\mathbf{n},
\end{aligned}$$

where θ is the polar angle of the field point $\mathbf{x}$ with respect to a polar axis along the direction of the oscillation of the charge $\mathbf{a}$. A very similar result is obtained for the radiation from an oscillating *dipole*, and the $\sin^2\theta$ dependence is characteristic of dipole radiation.

9.5 Fields Produced by a Moving Charged Particle: The General Case

It is easy to see that for *general* accelerated motion of a charged particle, we may write

$$\mathbf{S}_{\text{far}} = \frac{1}{\mu_0 c}\mathbf{E}_{\text{far}}\times[\mathbf{n}\times\mathbf{E}_{\text{far}}],$$

or since $\mathbf{n}\cdot\mathbf{E}_{\text{far}} = 0$,

$$\mathbf{S}_{\text{far}} = \frac{1}{\mu_0 c}|\mathbf{E}_{\text{far}}|^2\mathbf{n}.$$

The *power* radiated per unit solid angle is then

$$\frac{dP}{d\Omega} = \frac{1}{\mu_0 c}|R\mathbf{E}_{\text{far}}|^2.$$

If furthermore the motion of the particle is non-relativistic, $\beta << 1$, we have

$$R\mathbf{E}_{\text{far}} = \frac{q}{4\pi\epsilon_0}\frac{1}{c}[\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})],$$

so that

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{\mu_0 c}\Big(\frac{q}{4\pi\epsilon_0}\Big)^2\frac{1}{c^2}[\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})]^2 \\
&= \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c^3}|\dot{\mathbf{u}}|^2\sin^2\theta,
\end{aligned}$$

where θ is the angle between the dirction $\mathbf{n}$ of the field point and the instantaneous acceleration $\dot{\mathbf{u}}$ of the particle; and the radiation is polarised in the plane containing $\mathbf{n}$ and $\dot{\mathbf{u}}$. The total instantaneous power radiated is obtained by integration over angles, and using $\int d\Omega \sin^2\theta = 8\pi/3$ one obtains the *Larmor formula*:

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}|\dot{\mathbf{u}}|^2.$$

The Larmor formula has used $\beta << 1$, but if written in the form

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}\frac{1}{m^2}\left(\frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt}\right)$$

where $\mathbf{p} = m\mathbf{u}$ is the momentum of the charge, it generalises to

$$\begin{aligned} P &= -\frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}\frac{1}{m^2}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) \\ &= \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c}\gamma^6[(\dot{\boldsymbol{\beta}})^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2] \end{aligned}$$

which is the *Liénard formula*.

9.6 Motion in a Circle

Specialising to the case of motion in a circle, where

$$\left|\frac{d\mathbf{p}}{d\tau}\right| = \left|\gamma\frac{d\mathbf{p}}{dt}\right| = \gamma\omega|\mathbf{p}|.$$

If the energy loss per revolution is small,

$$\frac{1}{c}\frac{dE}{d\tau} << \left|\frac{d\mathbf{p}}{d\tau}\right|,$$

we have

$$\begin{aligned} P &= \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}\frac{1}{m^2}\gamma^2\omega^2|\mathbf{p}|^2 \\ &= \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c}\frac{1}{m^2}\gamma^2\omega^2\gamma^2\beta^2 m^2 \\ &= \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}c\beta^4\gamma^4\frac{1}{\rho^2} \end{aligned}$$

using that the radius $\rho = c\beta/\omega$ for motion in a circle. The energy lost in a single revolution is thus $\Delta E = P\times$ the time for a single revolution,

$$\begin{aligned} \Delta E &= P\frac{2\pi}{\omega} \\ &= P\,2\pi\,\frac{\rho}{c\beta} \\ &= \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{\rho}\beta^3\gamma^4 \\ &= \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0}\beta^3\left(\frac{E}{mc^2}\right)^4\frac{1}{\rho}. \end{aligned}$$

This *synchrotron radiation loss* can be very important for particles in accelerator or storage rings, and as we shall see, this is particularly true for electron machines. Expressing ΔE in GeV, and ρ in km, the formula above gives

$$\Delta E = 6\times 10^{-21}\frac{1}{\rho}\left(\frac{E}{mc^2}\right)^4.$$

Consider for example HERA, the collider at DESY. This machine has a circumference of 6336 m, so we may take the radius to be approximately 1 km. Positrons (or electrons) with an energy of 30 GeV are made to collide with protons with an energy of 820 GeV. For positrons $mc^2 \approx 0.5$ MeV, whilst for protons $mc^2 \approx 1$ GeV. With these data the formula for the energy loss per revolution gives

$$\Delta E \approx 80\text{MeV for the positrons,}$$

and only

$$\approx 2.6 \times 10^{-6}\text{MeV for the protons.}$$

The power drain that maintaining the energy of the circulating beams imposes is very significant for positron or electron machines. It is simple to derive

$$P(\text{watts}) \approx 10^9 \Delta E(\text{GeV}) J(\text{amps})$$

which with a current of 12 milliamps in the HERA positron beam gives around a Megawatt, or some 140 watts per metre! Of course in a machine such as the synchrotron source at Daresbury, the whole point of the device is to produce a brilliant source of radiation.