Chapter 4

Interacting fields

In the previous chapter, we saw how a free field can be quantised. More realistic fields will, however, have interactions, and quantising such fields will be the focus of this chapter. Unlike the former, this case cannot be treated exactly, but if the coupling strength is small, a perturbative scheme can be used. This would turn out to be similar to that introduced in Sec. 2.7.

One simple example of interacting fields is provided by QED, which describes electrons and photons. Recall that the Lagrangian for a free electron is given by (3.76). To include the electromagnetic interaction, we have to redefine the partial derivative

$$i\partial_\mu \to i\partial_\mu - qA_\mu \,, \tag{4.1}$$

where the charge is $q = -e$. This contributes an interaction term

$$\mathcal{L}_{\text{int}} = -q\bar{\psi}\gamma_\mu\psi A^\mu = e\, j \cdot A \,, \tag{4.2}$$

to the Lagrangian. The full Lagrangian is then given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \,, \tag{4.3}$$

where $\mathcal{L}_0$ is the sum of the non-interacting electron and photon Lagrangians.

A field can also interact with itself. For example, quantum chromodynamics (QCD) is like QED, except that A is non-Abelian. This gives rise to terms quadratic in A in the Lagrangian. The analogue of the photon is the gluon, and the analogue of electric charge is colour. While the photon does not carry any charge, the gluon carries colour. Since the gluon interacts with anything that carries colour, it in particular interacts with itself.

Another example of a self-interacting field is the scalar field ϕ with a quartic term in the Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} = \mathcal{L}_0 - \tfrac{1}{4!}\lambda\phi^4, \tag{4.4}$$

where $\mathcal{L}_0$ is the free Lagrangian introduced in (3.26). We require the coupling constant to satisfy $\lambda > 0$, so that the Hamiltonian H is bounded from below. This theory is known as ϕ^4 theory,[1] and it is important in electroweak theory, where ϕ plays the role of the Higgs field giving mass to the gauge field.

In this case, the field equation is

$$(\partial^2 + m^2)\phi = j\,, \tag{4.5}$$

where the source term is given by

$$j = \frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi} = -\frac{1}{3!}\lambda\phi^3. \tag{4.6}$$

Recall that the conjugate momentum is given by $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$, so the interaction does not change it. Thus, the canonical commutation relations are also unchanged. But the plane-wave expansion of ϕ in terms of a, $a^\dagger$ is no longer valid, so $[\phi(x), \phi(y)] \neq i\triangle(x-y)$.

4.1 Asymptotic conditions

In a scattering experiment, the states in the remote past and future are described by wave packets that are well-separated in space. For forces of finite range, the initial and final states are effectively the same as if the interaction is switched off:

$$\phi(x) \to \phi_{\substack{\text{in}\\\text{out}}}(x)\,, \qquad x^0 \to \mp\infty\,, \tag{4.7}$$

[1] Other interactions, like $\mathcal{L}_{\text{int}} = \frac{1}{3!}\lambda\phi^3$ can be used, but H is not bounded from below. One can also consider derivative terms, e.g., $\lambda\phi^2(\partial\phi)^2$ or higher powers of ϕ, but these theories turn out to be non-renormalisable.

where $\phi_{\rm in}$ and $\phi_{\rm out}$ are *free* fields satisfying the Klein–Gordon equation

$$(\partial^2 + m^2)\phi_{\substack{\rm in\\ \rm out}} = 0\,. \tag{4.8}$$

These are the boundary conditions imposed on the solution of the field equation. Hence, the general solution to (4.5) is[2]

$$\phi(x) = \phi_{\substack{\rm in\\ \rm out}}(x) + \int_{-\infty}^{\infty} {\rm d}^4 y \, \triangle_{\substack{\rm R\\ \rm A}}(x-y) j(y)\,. \tag{4.9}$$

Now, the in- and out-fields satisfy the covariant commutation relation:

$$[\phi_{\rm in}(x), \phi_{\rm in}(y)] = i\triangle(x-y) = [\phi_{\rm out}(x), \phi_{\rm out}(y)]\,. \tag{4.10}$$

This equation means there is a unitary operator S, independent of x, such that

$$\phi_{\rm out}(x) = S^{-1}\phi_{\rm in}(x) S\,. \tag{4.11}$$

S is known as the scattering or S-matrix.

Because $\phi_{\rm in}$ and $\phi_{\rm out}$ are free fields, there are the corresponding creation operators $a^\dagger_{\rm in}(\mathbf{k})$ and $a^\dagger_{\rm out}(\mathbf{k})$ satisfying

$$a^\dagger_{\rm out}(\mathbf{k}) = S^{-1} a^\dagger_{\rm in}(\mathbf{k}) S\,. \tag{4.12}$$

Applying the free-field results obtain in Sec. 3.2, we may then construct the states

$$|\mathbf{k}_1, \mathbf{k}_2, \ldots; \begin{matrix}{\rm in}\\ {\rm out}\end{matrix}\rangle = a^\dagger_{\substack{\rm in\\ \rm out}}(\mathbf{k}_1) a^\dagger_{\substack{\rm in\\ \rm out}}(\mathbf{k}_2) \cdots |0\rangle\,, \tag{4.13}$$

which are eigenstates of the corresponding in- and out-Hamiltonians:

$$H_{\substack{\rm in\\ \rm out}} = \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} k_0 \, a^\dagger_{\substack{\rm in\\ \rm out}}(\mathbf{k}) a_{\substack{\rm in\\ \rm out}}(\mathbf{k})\,. \tag{4.14}$$

Using (4.12), we see that these two Hamiltonians are related by $H_{\rm out} = S^{-1} H_{\rm in} S$. But

$$H \to H_{\substack{\rm in\\ \rm out}}\,, \qquad t \to \mp\infty\,, \tag{4.15}$$

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[2] In this expression, we may need to include $\mathrm{e}^{-\epsilon|y|^2}$ to get convergence, then set $\epsilon \to 0$. The assumption that letting $\epsilon \to 0$ at the end of the calculation of a physical quantity is called the adiabatic hypothesis.

and H is time-independent, which means that $H = H_{\rm in} = H_{\rm out}$, and so $S^{-1}HS = H$.

Assume there is a unique vacuum $|0\rangle$ with zero eigenvalue for H:

$$0 = H|0\rangle = S^{-1}HS|0\rangle\,. \tag{4.16}$$

Thus $H(S|0\rangle) = 0$, which means that $S|0\rangle$ is a constant multiple of $|0\rangle$: $S|0\rangle = c|0\rangle$. But

$$|c|^2\langle 0|0\rangle = \langle 0|S^\dagger S|0\rangle = \langle 0|0\rangle\,, \tag{4.17}$$

by the unitarity of S, so $c = \mathrm{e}^{i\delta}$ is a phase factor which has no observable consequences. We shall therefore set $c = 1$.

Now, the probability of a transition from an in-state α to an out-state β is

$$\begin{aligned} |\langle\beta, \text{out}|\alpha, \text{in}\rangle|^2 &= |\langle\beta, \text{in}|S|\alpha, \text{in}\rangle|^2 \\ &= |\langle\beta, \text{out}|S|\alpha, \text{out}\rangle|^2, \end{aligned} \tag{4.18}$$

where we have used $|\beta, \text{out}\rangle = S^{-1}|\beta, \text{in}\rangle$ (which follows from (4.12), (4.13), and the fact that $S|0\rangle = |0\rangle$). The unitarity of S corresponds to probability conservation. To see this, let the initial state be

$$|\,\cdot\,\rangle = \sum_\alpha \lambda_\alpha|\alpha, \text{in}\rangle\,, \quad \text{with} \quad \sum_\alpha |\lambda_\alpha|^2 = 1\,. \tag{4.19}$$

The probability of a transition to some final state is

$$\begin{aligned} 1 &= \sum_\beta |\langle\beta, \text{in}|S|\,\cdot\,\rangle|^2 \\ &= \sum_{\alpha,\alpha',\beta} \lambda^*_{\alpha'}\lambda_\alpha\langle\alpha', \text{in}|S^\dagger|\beta, \text{in}\rangle\langle\beta, \text{in}|S|\alpha, \text{in}\rangle \\ &= \sum_{\alpha,\alpha'} \lambda^*_{\alpha'}\lambda_\alpha\langle\alpha', \text{in}|S^\dagger S|\alpha, \text{in}\rangle\,. \end{aligned} \tag{4.20}$$

This is valid for all choices of $|\,\cdot\,\rangle$, if and only if $S^\dagger S = 1$. Similarly, $SS^\dagger = 1$ corresponds to the probability that, given any final state, there was some initial state leading to it.

4.2 Expression for S-matrix

The goal of this section is to find a perturbative expression for the scattering matrix S. Recall that, in the Heisenberg picture, we have

$$\phi(t,\mathbf{x}) = \mathrm{e}^{iH(t-t_0)}\phi(t_0,\mathbf{x})\mathrm{e}^{-iH(t-t_0)}, \tag{4.21}$$

relating a field ϕ at time t to one at earlier time t_0. Suppose we take $t_0 \to -\infty$, so that

$$\phi(x) = \lim_{t_0\to-\infty} \mathrm{e}^{iH(t-t_0)}\phi_{\mathrm{in}}(t_0,\mathbf{x})\mathrm{e}^{-iH(t-t_0)}. \tag{4.22}$$

Since ϕ_{in} is a free field, its time evolution in the Heisenberg picture:

$$\phi_{\mathrm{in}}(t_0,\mathbf{x}) = \mathrm{e}^{-iH_0^{\mathrm{in}}(t-t_0)}\phi_{\mathrm{in}}(t,\mathbf{x})\mathrm{e}^{iH_0^{\mathrm{in}}(t-t_0)}, \tag{4.23}$$

is determined by H_0^{in}, the non-interacting (free) part of H with ϕ replaced by ϕ_{in}. Substituting (4.23) into (4.22) then gives

$$\phi(x) = U^{-1}(t)\phi_{\mathrm{in}}(x)U(t)\,, \tag{4.24}$$

where the unitary operator $U(t)$ is given by

$$U(t) = \lim_{t_0\to-\infty} \mathrm{e}^{iH_0^{\mathrm{in}}(t-t_0)}\mathrm{e}^{-iH(t-t_0)}. \tag{4.25}$$

Note that (4.11) implies that $U(\infty) = S$.

Our next task is to determine U, from which we can deduce S. If we differentiate (4.25) with respect to t, we get

$$\dot{U}(t) = \lim_{t_0\to-\infty} i\mathrm{e}^{iH_0^{\mathrm{in}}(t-t_0)}(H_0^{\mathrm{in}} - H)\mathrm{e}^{-iH(t-t_0)}. \tag{4.26}$$

This equation can be written quite simply as

$$i\dot{U}(t) = H_{\mathrm{I}}(t)U(t)\,, \tag{4.27}$$

where

$$H_{\mathrm{I}}(t) = \lim_{t_0\to-\infty} \mathrm{e}^{iH_0^{\mathrm{in}}(t-t_0)}(H - H_0^{\mathrm{in}})\mathrm{e}^{-iH_0^{\mathrm{in}}(t-t_0)}. \tag{4.28}$$

(The subscript I indicates that the interaction Hamiltonian is in the so-called interaction picture.) Since both H and H_0^{in} are independent of t, we can evaluate them at $t = t_0$. In this case,

$$H - H_0^{\text{in}} = H_{\text{int}}(\phi_{\text{in}}(t_0))\,. \tag{4.29}$$

Let us assume there are *no* derivatives in H_{int}, so it can only be a polynomial in ϕ. For example,

$$\begin{aligned} H_{\text{int}} &= \frac{1}{4!}\lambda\phi^4 \qquad \text{for } \phi^4\text{-theory;} \\ H_{\text{int}} &= \frac{1}{3!}\lambda\phi^3 \qquad \text{for } \phi^3\text{-theory.} \end{aligned} \tag{4.30}$$

Then substituting (4.29) into (4.28), and using (4.23) for each power of ϕ in H, we have

$$H_{\text{I}}(t) = H_{\text{int}}(\phi_{\text{in}}(t))\,. \tag{4.31}$$

We may now integrate (4.27) to obtain an equation for U:

$$U(t) = 1 - i\int_{-\infty}^{t} \mathrm{d}t_1\, H_{\text{I}}(t_1)U(t_1)\,. \tag{4.32}$$

This can be solve iteratively. Taking the limit $t \to \infty$, we have the infinite sum

$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{t_1}\mathrm{d}t_2 \int_{-\infty}^{t_2}\mathrm{d}t_3 \cdots \int_{-\infty}^{t_{n-1}}\mathrm{d}t_n\, H_{\text{I}}(t_1)H_{\text{I}}(t_2)\cdots H_{\text{I}}(t_n)\,. \tag{4.33}$$

Using the properties of the step function $\theta(t)$, this equation can be rewritten as

$$\begin{aligned} S &= \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{\infty}\mathrm{d}t_2 \cdots \int_{-\infty}^{\infty}\mathrm{d}t_n\, \theta(t_1 - t_2)\cdots\theta(t_{n-1} - t_n)H_{\text{I}}(t_1)\cdots H_{\text{I}}(t_n) \\ &= \sum_{n=0}^{\infty}(-i)^n \sum_{\substack{\text{perm} \\ i\to\alpha_i}} \frac{1}{n!}\int_{-\infty}^{\infty}\mathrm{d}t_1 \cdots \int_{-\infty}^{\infty}\mathrm{d}t_n\, \theta(t_{\alpha_1} - t_{\alpha_2})\cdots\theta(t_{\alpha_{n-1}} - t_{\alpha_n})\times \\ &\qquad\qquad \times H_{\text{I}}(t_{\alpha_1})\cdots H_{\text{I}}(t_{\alpha_n})\,, \end{aligned} \tag{4.34}$$

where in the second line, we have permuted the n indices α_i, and had to divide by $n!$ as a result. Now, the integrand can be turned into a time-ordered product by an identity similar to (3.102):

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}\mathrm{d}t_1 \cdots \int_{-\infty}^{\infty}\mathrm{d}t_n\, \mathrm{T}\big\{H_{\text{I}}(t_1)H_{\text{I}}(t_2)\cdots H_{\text{I}}(t_n)\big\} \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \cdots \int \mathrm{d}^4x_n\, \mathrm{T}\big\{\mathcal{L}_{\text{int}}^{\text{in}}(x_1)\mathcal{L}_{\text{int}}^{\text{in}}(x_2)\cdots\mathcal{L}_{\text{int}}^{\text{in}}(x_n)\big\}\,, \end{aligned} \tag{4.35}$$

where $\mathcal{L}_{\rm int}^{\rm in}$ is the interaction part of the Lagrangian $\mathcal{L}$, with ϕ replaced by $\phi_{\rm in}$. This is our desired expression for the scattering matrix. (In fact, it can be shown that (4.35) is valid even for interactions involving derivatives.)

Note that for time-like $x_i - x_j$, Lorentz transformations preserve the sign of $x_i^0 - x_j^0$, i.e., the chronological order of events. But this is not necessarily true for space-like separations, although in this case,

$$[\mathcal{L}_{\rm int}^{\rm in}(x_i), \mathcal{L}_{\rm int}^{\rm in}(x_j)] = 0\,, \tag{4.36}$$

so the time-ordered product is still Lorentz invariant. It is often convenient to write (4.35) formally as

$$S = \mathrm{T}\,\exp i \int \mathrm{d}^4x\, \mathcal{L}_{\rm int}^{\rm in}(x)\,. \tag{4.37}$$

4.3 Elements of the S-matrix

Having obtained an expression for the S-matrix, we would now like to calculate its matrix elements:

$$\langle k_1', k_2', \ldots, k_n'; \mathrm{in}|S|k_1, k_2, \ldots, k_n; \mathrm{in}\rangle\,, \tag{4.38}$$

term by term in the perturbative expansion. To do so, it is advantageous to write each term in S as a sum of normal-ordered products of the form:

$$(\text{creation})(\text{annihilation})\,. \tag{4.39}$$

This greatly simplifies the calculation: if all the k' are different from all the k, the only non-zero contributions come from terms whose annihilation operators are precisely those which turn $|\cdot\rangle$ into $|0\rangle$, and whose creation operators are those which turn $\langle\cdot|$ into $\langle 0|$. But since each term in the perturbative expansion (4.35) is a time-ordered product

$$\mathrm{T}\{A(x_1)B(x_2)C(x_3)\cdots\}\,, \tag{4.40}$$

we need a general formula to expand this in terms of normal-ordered products. This formula is known as *Wick's theorem*.

From the definition of the normal-ordered product, we have for two field operators $A \equiv A(x_1) = A^{(+)}(x_1) + A^{(-)}(x_1)$ and $B \equiv B(x_2) = B^{(+)}(x_2) + B^{(-)}(x_2)$ that

$$AB - :AB: = \begin{cases} \{A^{(+)}, B^{(-)}\} & \text{for two fermion fields;} \\ [A^{(+)}, B^{(-)}] & \text{otherwise.} \end{cases} \tag{4.41}$$

The right-hand side is a c-number that does not involve creation and annihilation operators. Taking the vacuum expectation value of this equation tells us that the right-hand side is given by $\langle 0|AB|0\rangle$. Hence

$$AB =: AB : + \langle 0|AB|0\rangle \,. \tag{4.42}$$

Since $: AB := \pm : BA :$ (minus sign for two fermion fields and plus sign for all other cases), we have in particular that

$$\mathrm{T}\{AB\} =: AB : + \langle 0|\mathrm{T}\{AB\}|0\rangle \,. \tag{4.43}$$

It is convenient to introduce the notation

$$\overbrace{A(x_1)B(x_2)} \equiv \langle 0|\mathrm{T}\{A(x_1)B(x_2)\}|0\rangle \,, \tag{4.44}$$

which is known as the *contraction* of A and B. At this stage, we recall the following results derived in Chapter 3:

$$\begin{aligned}
\overbrace{\phi(x_1)\phi(x_2)} &= i\triangle_{\mathrm{F}}(x_1 - x_2) \,, \\
\overbrace{\phi(x_1)\phi^\dagger(x_2)} = \overbrace{\phi^\dagger(x_2)\phi(x_1)} &= i\triangle_{\mathrm{F}}(x_1 - x_2) \,, \\
\overbrace{\psi(x_1)\bar{\psi}(x_2)} = -\overbrace{\bar{\psi}(x_2)\psi(x_1)} &= iS_{\mathrm{F}}(x_1 - x_2) \,, \\
\overbrace{A^\mu(x_1)A^\nu(x_2)} &= ig^{\mu\nu}\triangle_{\mathrm{F}}(x_1 - x_2)\big|_{m=0} \,,
\end{aligned} \tag{4.45}$$

i.e., the contraction of two fields of the same type gives its corresponding Feynman propagator.

We would now like to generalise (4.43) to several operators $A = A(x_1)$, $B = B(x_2)$, $C = C(x_3)$, ... Wick's theorem (not proved here) states that

$$\begin{aligned}
\mathrm{T}\{ABCD\cdots WXYZ\} =& : ABCD\cdots WXYZ : \\
&+ : \overbrace{ABC\cdots YZ} : + : A\overbrace{BC\cdots YZ} : + \cdots + : ABC\cdots \overbrace{YZ} : \\
&+ : \overbrace{AB}\overbrace{CD}\cdots YZ : + \cdots + : AB\cdots \overbrace{WX}\overbrace{YZ} : \\
&+ \cdots \,,
\end{aligned} \tag{4.46}$$

where the first, second and third lines represent all terms with no, one and two contractions, and so on. Each term on the right-hand side is a normal-ordered product, which is the desired result.

To further simplify each of these terms, we need to use the relation:

$$:\underbrace{ABCDEF\cdots JKLM}\cdots: = (-1)^P \underbrace{AK}\,\underbrace{BC}\,\underbrace{EL}\cdots : DF\cdots JM\cdots :, \tag{4.47}$$

where P is the number of interchanges of neighbouring fermion operators required to change the order $ABC\cdots$ to $AKB\cdots$.

As an application of this formalism, we shall consider QED. As mentioned at the beginning of this chapter, the interaction is given by $\mathcal{L}_{\rm int} = -q\bar{\psi}\gamma^\mu\psi A_\mu$. We shall omit the γ^μ and the index μ in what follows, for notational simplicity. For the $n = 2$ term in S, we need to evaluate

$$\mathrm{T}\{\bar{\psi}_{\rm in}(x)\psi_{\rm in}(x)A_{\rm in}(x)\bar{\psi}_{\rm in}(y)\psi_{\rm in}(y)A_{\rm in}(y)\}\,. \tag{4.48}$$

Recall from (3.79) that the free-field plane-wave expansions of ψ and $\bar{\psi}$ take the form

$$\begin{aligned}\psi_{\rm in} &= \int_{\substack{\text{annihilates}\\ \text{electrons }(+)}} au \;+\; \int_{\substack{\text{creates}\\ \text{positrons }(-)}} b^\dagger v\,,\\ \bar{\psi}_{\rm in} &= \int_{\substack{\text{annihilates}\\ \text{positrons }(+)}} b\bar{v} \;+\; \int_{\substack{\text{creates}\\ \text{electrons }(-)}} a^\dagger \bar{u}\,,\end{aligned} \tag{4.49}$$

where the first term on the right-hand side has positive frequency $(+)$, and the second term negative frequency $(-)$. These terms have the interpretation of creating/annihilating electrons/positrons, as indicated. Similarly, the gauge field can be expanded as

$$A_{\rm in} = \underset{\substack{\text{annihilates}\\ \text{photons}}}{A^{(+)}} + \underset{\substack{\text{creates}\\ \text{photons}}}{A^{(-)}}\,.$$

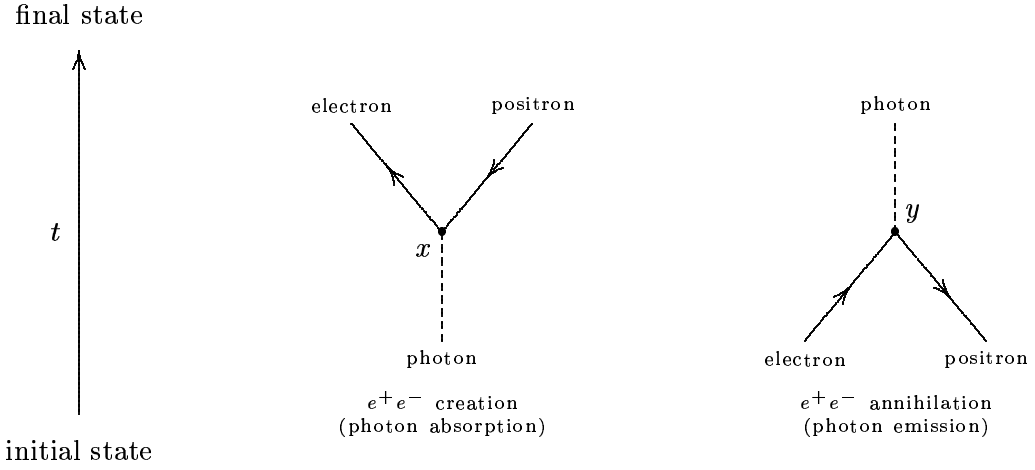
Let us first consider terms with no contractions, e.g.,

$$\bar{\psi}^{(-)}(x)\psi^{(-)}(x)A^{(-)}(y)\bar{\psi}^{(+)}(y)\psi^{(+)}(y)A^{(+)}(x)\,. \tag{4.50}$$

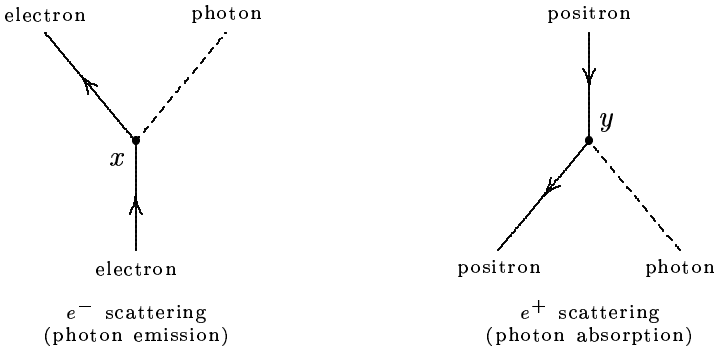
This operator only gives a non-zero contribution for the state:

$$\langle \underset{\rm ele}{x}, \underset{\rm pos}{x}, \underset{\rm photon}{y} \;|\qquad |\; \underset{\rm pos}{y}, \underset{\rm ele}{y}, \underset{\rm photon}{x} \rangle\,, \tag{4.51}$$

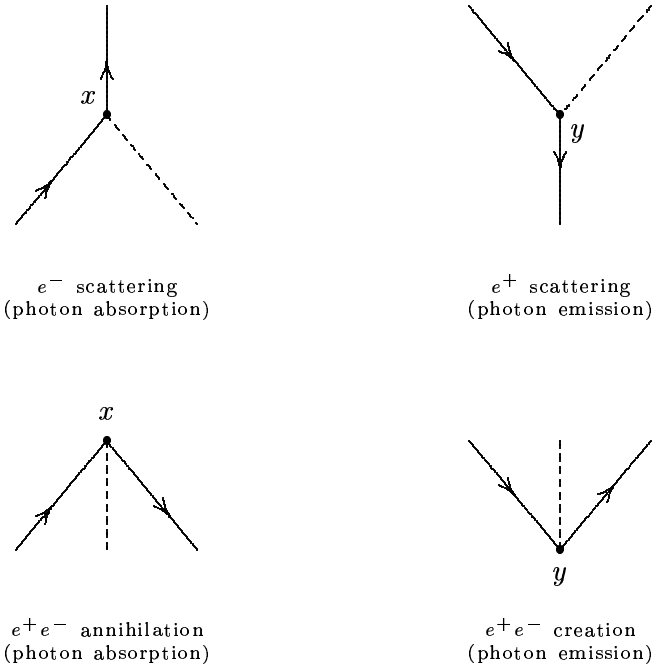
as can be seen by considering the effect of the annihilation operators in $\bar{\psi}^{(+)}$, $\psi^{(+)}$ and $A^{(+)}$, and the effect of the creation operators in $\bar{\psi}^{(-)}$, $\psi^{(-)}$ and $A^{(-)}$ (which become annihilation operators when the adjoint is taken). Thus (4.50) describes the transition of a photon at x into an electron–positron pair, and the transition of a positron and electron at y into a photon. Feynman introduced pictures to describe this process:



where the dotted lines represent photons and the solid lines denote electrons/positrons. Here, a positron is regarded as an electron propagating backwards in time, hence the arrow directions. Similarly, the term $\bar{\psi}^{(-)}(x)\psi^{(-)}(y)A^{(-)}(x)\bar{\psi}^{(+)}(y)\psi^{(+)}(x)A^{(+)}(y)$ gives the diagram:



We may get all the contributions corresponding to no contractions by swinging round the lines. The other independent Feynman diagrams are



e^- scattering (photon absorption)

e^+ scattering (photon emission)

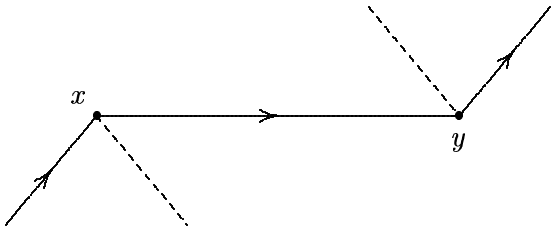
e^+e^- annihilation (photon absorption)

e^+e^- creation (photon emission)

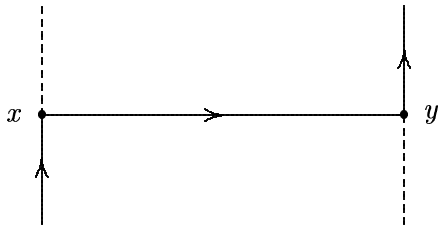
We consider now a term with one contraction, e.g.,

$$\bar{\psi}^{(-)}(y)\underbrace{\psi^{(-)}(y)A^{(-)}(y)\bar{\psi}^{(+)}}(x)\psi^{(+)}(x)A^{(+)}(x)$$
$$= iS_{\rm F}(y-x)\,\bar{\psi}^{(-)}(y)A^{(-)}(y)\psi^{(+)}(x)A^{(+)}(x)\,, \qquad (4.52)$$

where we have used (4.45). This describes a diagram with an internal electron propagating from x to y:

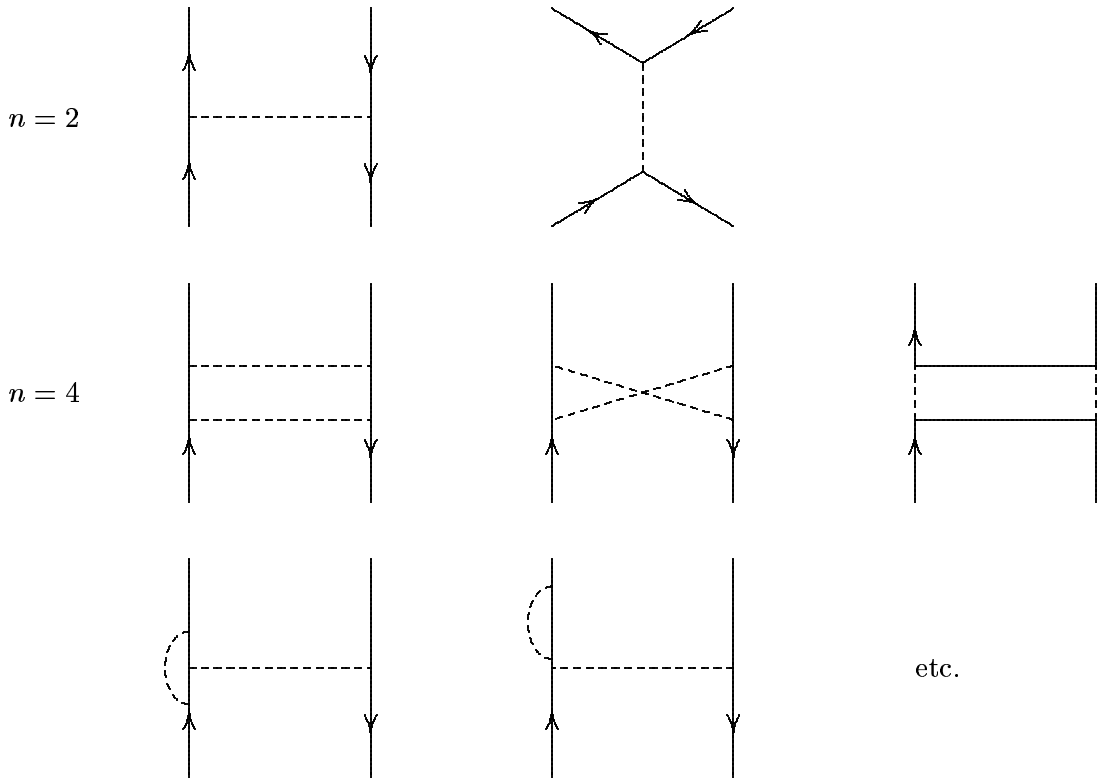


Other diagrams can be obtained by swinging round the external lines, for example,

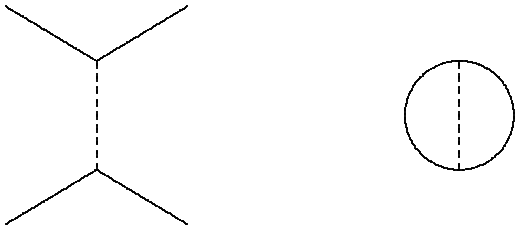


One simplification is to omit the other topologically identical diagrams obtained just by permuting the integration variables $x, y, \ldots$, and omitting the $\frac{1}{n!}$ in (4.35).

It is now clear that each interaction vertex has one photon line, and one directed fermion line entering and leaving it. Using this general rule, we can draw the diagrams corresponding to any physical process. For example, for electron–positron scattering to second and fourth order, we have the diagrams:



The internal lines are said to represent *virtual particles*. We do not include disconnected graphs, such as



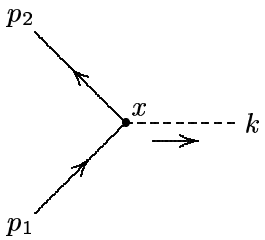
because if one sums over all such graphs, the result is then

$$\langle S\rangle = \langle S\rangle_{\text{connected}}\langle 0|S|0\rangle\,, \tag{4.53}$$

where $\langle 0|S|0\rangle$ represents all those 'vacuum' graphs which do not have external legs. By (4.17), the latter is a phase factor $\mathrm{e}^{i\delta}$ which disappears when one calculates $|\langle S\rangle|$. (If one did calculate δ, it would be infinite!)

Conversely, given a particular diagram, it is possible to write down the corresponding matrix element. For a graph with n vertices, label them x, y, ... and assign to each a Lorentz index $\mu, \nu, \ldots$. Then we must include: $-iq\gamma^\mu$ for each vertex; $-ig^{\mu\nu}D_{\mathrm{F}}(x-y)$ for each internal photon line joining x, y; $iS_{\mathrm{F}}(x-y)$ for each internal electron line directed from x to y; as well as wave functions for the external lines: e.g., $u(\mathbf{p})\mathrm{e}^{-ip\cdot x}$ for an initial electron annihilated at x and with momentum p.

We then integrate over $x, y, \ldots$. This is straightforward if we use the Fourier integrals for $D_{\mathrm{F}}(x-y)$ and $S_{\mathrm{F}}(x-y)$, so that the $x, y, \ldots$ dependence comes only in exponentials. For example,



$$\int \mathrm{d}^4x\,\mathrm{e}^{-ip_1\cdot x+ip_2\cdot x+ik\cdot x} = (2\pi)^4\delta^{(4)}(p_2+k-p_1)\,, \tag{4.54}$$

giving energy-momentum conservation at each vertex. The momenta remaining are either those of external particles, or integration variables in the Fourier integrals for the virtual-line propagators.

The Feynman rules for matrix elements of S are therefore

1. $-iq\gamma^\mu(2\pi)^4\delta^{(4)}$(energy-momentum) at each vertex.

2. Each internal photon line gives

$$-\frac{ig^{\mu\nu}}{(2\pi)^4}\frac{1}{k^2+i\epsilon} \qquad \mu \bullet \xrightarrow[k]{} \bullet \nu \tag{4.55}$$

3. Each internal electron line gives

$$\frac{i}{(2\pi)^4}\frac{\gamma\cdot p+m}{p^2-m^2+i\epsilon} \qquad \xrightarrow[p]{} \tag{4.56}$$

4. Spinor wave-functions for the external lines:

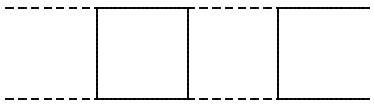
   $u(\mathbf{p})$ for initial-state electron of momentum $\mathbf{p}$;

   $\bar{v}(\mathbf{p})$ " " " positron " " ";

   $\bar{u}(\mathbf{p})$ " final " electron " " "; and

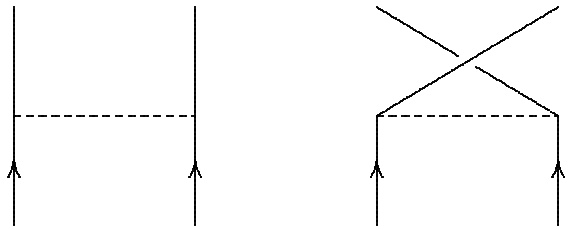
   $v(\mathbf{p})$ " " " positron " " ".

5. For each directed line, start at the end to which the arrow points, multiply the spinors, the gamma matrices, and the propagators, in order along the line. For each closed fermion loop, take the trace and include the factor -1. An example of such a loop is



6. Integrate over all internal momenta.

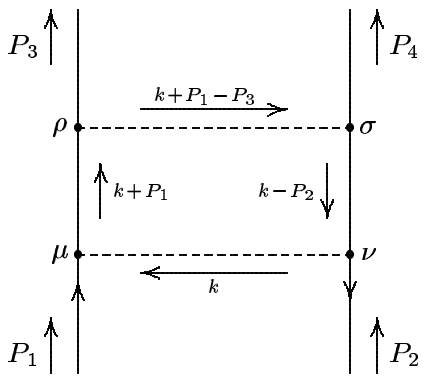
The overall sign may still be wrong, because of the $(-1)^P$ factors in Wick's theorem, coming from (4.47). But because we will finally take the modulus of the S-matrix element, we need only to know that graphs which differ in the interchange of identical fermions differ by a minus sign. For example, the following two graphs have opposite signs:



Now, because energy-momentum is conserved at each vertex, it is conserved in the overall scattering process. It is convenient to write

$$\langle f|S|i\rangle = \langle f|1 + iT(2\pi)^4\delta^{(4)}(P_i - P_f)|i\rangle\,, \tag{4.57}$$

where the 1 corresponds to the $n = 0$ term in the expansion and contributes only if $|i\rangle = |f\rangle$. We take into account conservation at each vertex by introducing loop momenta. For example, consider the diagram:



with $P_1 + P_2 = P_3 + P_4$. To calculate T, we first count the factors $(2\pi)^4$ and i. In the above diagram, we have a factor of $-i(2\pi)^4$ for each vertex; $\frac{i}{(2\pi)^4}$ for each propagator; and $\frac{1}{i(2\pi)^4}$ to get from S to T. Hence, the total contribution is $-\frac{i}{(2\pi)^4}$, and

$$\begin{aligned} T = -\frac{i}{(2\pi)^4}(-e)^4 \int \mathrm{d}^4k \left\{ \bar{u}(P_3)\gamma^\rho \frac{\gamma\cdot(P_1+k)+m}{(P_1+k)^2-m^2+i\epsilon}\gamma^\mu u(P_1) \right\} \times \\ \times \left\{ \bar{v}(P_2)\gamma^\nu \frac{\gamma\cdot(k-P_2)+m}{(k-P_2)^2-m^2+i\epsilon}\gamma^\sigma v(P_4) \right\} \times \\ \times \left\{ \frac{-g_{\mu\nu}}{k^2+i\epsilon} \right\} \left\{ \frac{-g_{\rho\sigma}}{(k+P_1-P_3)^2+i\epsilon} \right\}, \end{aligned} \tag{4.58}$$

where we integrate over the loop momenta k. T is known as the *scattering amplitude* (also called the T-matrix), and it contains all the information for a real scattering transition (i.e., when $|i\rangle \neq |f\rangle$).

4.4 ϕ^4 theory

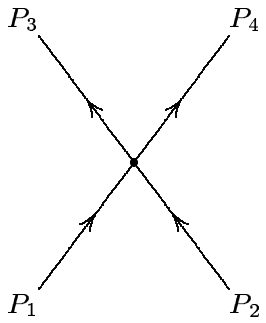
We shall now briefly digress to examine another example of an interacting theory, namely ϕ^4 theory. The (normal-ordered) interaction Lagrangian is

$$\mathcal{L}_{\mathrm{int}} =: -\tfrac{1}{4!}\lambda\phi^4 : \,. \tag{4.59}$$

This represents a vertex where four scalar field lines meet. For example, consider the scattering

$$P_1 + P_2 \to P_3 + P_4\,. \tag{4.60}$$

The lowest-order ($n = 1$) process is

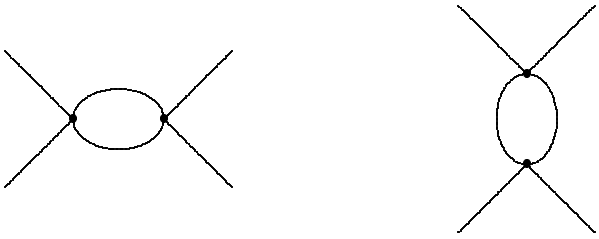


Note that there are 4! ways of identifying the four factors ϕ in $i \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{int}}^{\mathrm{in}}$ with the external legs of this diagram. This cancels the factor of $\frac{1}{4!}$ in $\mathcal{L}_{\mathrm{int}}^{\mathrm{in}}$. It can be checked that the S-matrix element for this process is

$$\langle f|S|i\rangle = -i\lambda(2\pi)^4\delta^{(4)}(P_1 + P_2 - P_3 - P_4)\,, \tag{4.61}$$

so the amplitude is just $\langle f|T|i\rangle = -\lambda$.

Some diagrams of order $n = 2$ are



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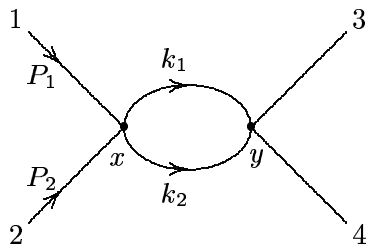
There are, in fact, 8! possibilities. The number of diagrams increases rapidly for higher orders.

It is now straightforward to write down the Feynman rules for constructing matrix elements of S:

1. $-i\lambda(2\pi)^4\delta^{(4)}$(energy-momentum) at each vertex.
2. $\frac{i}{(2\pi)^4}\frac{1}{p^2-m^2+i\epsilon}$ for each internal line.
3. Integrate over internal momenta.

Because the four legs at each vertex are identical, many diagrams are, in fact, topologically equivalent. To simplify the calculations, we may just consider topologically distinct diagrams and include a certain weight factor in the end. Two examples:

(a)



'fish' diagram

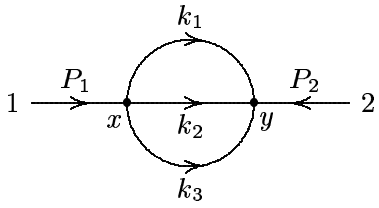
There are four ways to attach the first leg of vertex x to 1, and three ways to attach the second leg to 2. Similarly for vertex y with respect to 3 and 4. Finally, there are two ways to join up the remaining four internal legs. Hence, the weight factor is

$$\left(\frac{1}{4!}\right)^2 4\cdot 3\cdot 4\cdot 3\cdot 2 = \frac{1}{2}\,. \tag{4.62}$$

Applying the Feynman rules then gives the amplitude

$$\begin{aligned} &-\frac{i}{(2\pi)^4}(-\lambda)^2\frac{1}{2}\int \mathrm{d}^4k_1 \int \mathrm{d}^4k_2\, \frac{\delta^{(4)}(k_1+k_2-P_1-P_2)}{(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)} \\ &\quad = -\frac{i}{(2\pi)^4}\frac{\lambda^2}{2}\int \mathrm{d}^4k\, \frac{1}{(k^2-m^2+i\epsilon)((k-P_1-P_2)^2-m^2+i\epsilon)}\,. \end{aligned} \tag{4.63}$$

(b)



'setting-sun' diagram

There are four ways to attach the first leg of vertex x to 1, and four ways to attach the first leg of vertex y to 2. Furthermore, there are three ways to attach the second leg of vertex x to y, two ways to attach the third leg of vertex x to y. Hence, the weight factor is

$$\left(\frac{1}{4!}\right)^2 4\cdot 4\cdot 3\cdot 2 = \frac{1}{6}\,. \tag{4.64}$$

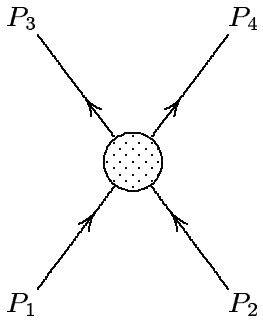
The amplitude is therefore

$$\frac{1}{(2\pi)^8}(-\lambda)^2\frac{1}{6}\int \mathrm{d}^4k_1 \int \mathrm{d}^4k_2 \int \mathrm{d}^4k_3\, \frac{\delta^{(4)}(P_1-k_1-k_2-k_3)\delta^{(4)}(P_2+k_1+k_2+k_3)}{(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)(k_3^2-m^2+i\epsilon)}\,. \tag{4.65}$$

4.5 Calculation of cross-section

Having learnt how to compute the amplitude of a scattering process, it remains to translate it into something that can be measured experimentally. The quantity of most interest is the differential cross-section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$, and in this section, we shall show how it is related to the amplitude T.

Now, in a scattering process, Lorentz invariance implies that the amplitude $T = T(s,t)$ depends only on two variables, namely



$$\begin{aligned} s &\equiv (P_1 + P_2)^2 = (P_3 + P_4)^2, \\ t &\equiv (P_1 - P_3)^2 = (P_2 - P_4)^2. \end{aligned} \tag{4.66}$$

Because $P_1^2 = P_2^3 = P_3^2 = P_4^2 = m^2$, all other Lorentz invariants can be expressed in terms of s and t. Another variable that is useful is

$$u \equiv (P_1 - P_4)^2 = (P_2 - P_3)^2, \tag{4.67}$$

which satisfies $s + t + u = 4m^2$. s, t and u are, of course, the well-known Mandelstam variables.

It is useful to specialise to a certain Lorentz frame when doing calculations. Two commonly used choices are the *laboratory* frame where the target particles are at rest, and the *centre-of-mass* frame. We shall consider the latter here. In this case, the total 3-momentum vanishes, so we may write $P_1 = (E_1, \mathbf{P})$ and $P_2 = (E_2, -\mathbf{P})$. This implies that $E_1 = \sqrt{m^2 + \mathbf{P}^2} = E_2$, and so the total centre-of-mass energy is

$$\sqrt{s} = E_1 + E_2 \geq 2m\,. \tag{4.68}$$

Also, we have $P_3 = (E_3, \mathbf{P}')$ and $P_4 = (E_4, -\mathbf{P}')$, where

$$E_3 = E_4 = \tfrac{1}{2}\sqrt{s}\,, \qquad |\mathbf{P}| = |\mathbf{P}'|\,. \tag{4.69}$$

The momentum transfer of the reaction is given by $t = -(\mathbf{P} - \mathbf{P}')^2 \leq 0$.

The probability of a transition from an initial state $|i\rangle$ to the final state $|f\rangle$ can be written as

$$|\langle f|S|i\rangle|^2 = |T(s,t)|^2(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4)(2\pi)^4\delta^{(4)}(0)\,. \tag{4.70}$$

But note that

$$(2\pi)^4\delta^{(4)}(0) = \int \mathrm{d}^4x\,\mathrm{e}^{ip\cdot x}\Big|_{p=0} = VT\,, \tag{4.71}$$

i.e., volume $\times$ time. Although this quantity is infinite (because V obviously is), we may put the system into a large but finite box of volume V. Because the volume would cancel out in the end, it does not change the final result.

Let the transition probability *rate* per unit volume to all possible final states be denoted by Γ. This is obtained from (4.70) by dividing by T and V, and integrating over final states:

$$\begin{aligned}\Gamma &= \int \frac{\mathrm{d}^3\mathbf{P}_3}{(2\pi)^3 2E_3}\int \frac{\mathrm{d}^3\mathbf{P}_4}{(2\pi)^3 2E_4}|T(s,t)|^2(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4)\\ &= \frac{1}{(2\pi)^2}\int \mathrm{d}^3\mathbf{P}_3\,\frac{1}{2E_3}\frac{1}{2E_4}|T(s,t)|^2\delta(E_1+E_2-E_3-E_4)\,.\end{aligned} \tag{4.72}$$

This expression can be further simplified if we write $\mathrm{d}^3\mathbf{P}_3 = |\mathbf{P}'|^2\mathrm{d}|\mathbf{P}'|\,\mathrm{d}\Omega$ in terms of the solid angle Ω, and perform the $\mathrm{d}|\mathbf{P}'|$ integration using the delta-function identity:

$$\int f(x)\delta[g(x)]\,\mathrm{d}x = \int f(x)\delta[g(x)]\left(\frac{\partial x}{\partial g}\right)\mathrm{d}g = \left[\frac{f(x)}{\partial g/\partial x}\right]_{g=0}\,. \tag{4.73}$$

Then (4.72) becomes, in the c.o.m. frame,

$$\begin{aligned}\Gamma &= \frac{1}{4\pi^2}\int |\mathbf{P}'|^2\mathrm{d}\Omega\,\frac{1}{2E_3}\frac{1}{2E_4}|T(s,t)|^2\frac{1}{\frac{\partial}{\partial|\mathbf{P}'|}2\sqrt{m^2+\mathbf{P}'^2}}\\ &= \frac{1}{16\pi^2}\int \mathrm{d}\Omega\,\frac{|\mathbf{P}'|}{\sqrt{s}}|T(s,t)|^2.\end{aligned} \tag{4.74}$$

To convert this to a cross-section, we have to divide by the incident flux $\mathcal{F}$, by definition.

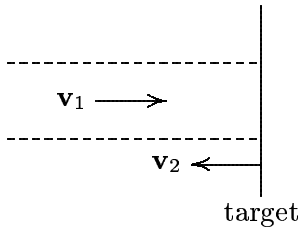
Now, the particle-flux of a momentum state $|P\rangle$ is given by

$$\langle P|P\rangle = 2P^0(2\pi)^3\delta^{(3)}(0) = 2P^0V\,, \tag{4.75}$$

corresponding to a particle-density of $2P^0$ (c.f. (3.61)). We may think of the P_2 particles as the 'target' particles, in which case there are $2E_2$ of them per unit volume. The number

of P_1 particles incident, per unit time, on a unit area of the plane that moves with the target particles, is

$2E_1 v_{\rm rel}$,



where the relative velocity of the P_1 and P_2 particles is

$$v_{\rm rel} = |\mathbf{v}_1 - \mathbf{v}_2| = \left| \frac{\mathbf{P}_1}{E_1} - \frac{\mathbf{P}_2}{E_2} \right| . \tag{4.76}$$

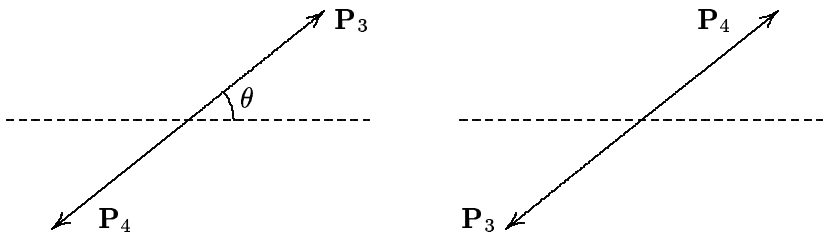
Thus, the flux factor is

$$\mathcal{F} = 2E_1 2E_2 v_{\rm rel} = 2E_1 2E_2 \left| \frac{\mathbf{P}_1}{E_1} - \frac{\mathbf{P}_2}{E_2} \right| = 4|\mathbf{P}|\sqrt{s} = 2\sqrt{s(s-4m^2)} \simeq 2s\,, \tag{4.77}$$

when $s \gg 4m^2$, i.e., for high-energy scattering. In this case, we also have $|\mathbf{P}'| \simeq \frac{1}{2}\sqrt{s}$. Substituting this into (4.74), and dividing by $\mathcal{F}$, we obtain the differential cross-section

$$\frac{{\rm d}\sigma}{{\rm d}\Omega} = \frac{1}{64\pi^2 s} |T(s,t)|^2 . \tag{4.78}$$

This formula is also valid for unequal-mass particles, provided $\sqrt{s} \gg$ masses. To get the total cross-section σ, we have to integrate over the solid angle Ω. But for identical particles $|\mathbf{P}_3, \mathbf{P}_4\rangle = \pm|\mathbf{P}_4, \mathbf{P}_3\rangle$, so we should only integrate over half the solid angle available to avoid overcounting. In this case, the diagrams



should not be both included.

Another useful form of the differential cross-section (4.78) can be obtained, if we observe that

$$t = (P_1 - P_3)^2 = -(\mathbf{P}^2 + \mathbf{P}'^2 - 2\mathbf{P}\cdot\mathbf{P}') = -\tfrac{1}{2}s(1-\cos\theta)\,, \tag{4.79}$$

for $\sqrt{s} \gg m$. Then,

$$\int \mathrm{d}\Omega = \int \mathrm{d}\phi\, \mathrm{d}(\cos\theta) = 2\pi \int \mathrm{d}(\cos\theta) = \frac{4\pi}{s} \int \mathrm{d}t\,, \tag{4.80}$$

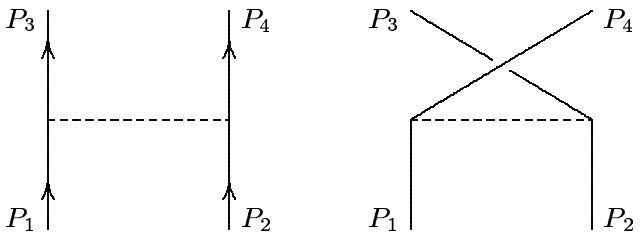
where we have used the fact that $|T|$ is independent of ϕ. Hence, (4.78) can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi s^2} |T(s,t)|^2. \tag{4.81}$$

This expression has the advantage of being manifestly Lorentz invariant.

4.6 Scattering of spin-$\frac{1}{2}$ particles

Consider $e^- e^-$ scattering to lowest non-trivial order ($n = 2$):

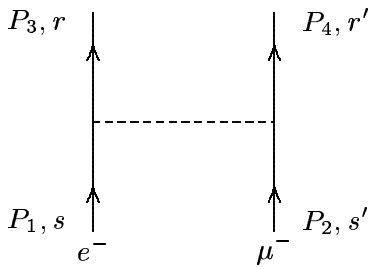


Using the Feynman rules for QED, the amplitude can be read off to be

$$T = e^2 \left\{ \frac{\bar{u}(P_3)\gamma^\mu u(P_1)\bar{u}(P_4)\gamma_\mu u(P_2)}{(P_3 - P_1)^2} - \frac{\bar{u}(P_4)\gamma^\mu u(P_1)\bar{u}(P_3)\gamma_\mu u(P_2)}{(P_4 - P_1)^2} \right\}. \tag{4.82}$$

The difference in sign between the two terms comes from interchanging the two electrons (fermions) $3 \leftrightarrow 4$.

For simplicity, we first consider electron–muon scattering, for which the second diagram is absent. If the spin states of the initial and final electron and muon are not known, we must *average* the transition probability over the possible initial spin states, and *sum* over the final ones. If we introduce the following notation for the spin states:



then we have

$$|T|^2 \to \tfrac{1}{4}\sum_{\text{spins}} T^*T = \frac{e^4}{4}\frac{1}{(P_3-P_1)^4}\sum_{r,r',s,s'}\big(\bar{u}_r(P_3)\gamma^\mu u_s(P_1)\big)^*\big(\bar{U}_{r'}(P_4)\gamma_\mu U_{s'}(P_2)\big)^*\times \\ \times\big(\bar{u}_r(P_3)\gamma^\nu u_s(P_1)\big)\big(\bar{U}_{r'}(P_4)\gamma_\nu U_{s'}(P_2)\big)\,, \quad (4.83)$$

where we write the muon spinor wave functions as U and $\bar{U}$ to distinguish them from the electron's. But note that

$$\begin{aligned}\sum_{r,s}\big(\bar{u}_r(P_3)\gamma^\mu u_s(P_1)\big)^*(\bar{u}_r(P_3)\gamma^\nu u_s(P_1)) &= \sum_{r,s}\big(u_s^\dagger(P_1)\gamma^{\mu\,\dagger}\gamma^0 u_r(P_3)\big)\big(\bar{u}_r(P_3)\gamma^\nu u_s(P_1)\big)\\ &= \sum_{r,s}\big(\bar{u}_s(P_1)\gamma^\mu u_r(P_3)\big)\big(\bar{u}_r(P_3)\gamma^\nu u_s(P_1)\big)\\ &= \sum_s \bar{u}_s(P_1)\gamma^\mu 2m\Lambda_+(P_3)\gamma^\nu u_s(P_1)\\ &= \sum_s \mathrm{Tr}\left(u_s(P_1)\bar{u}_s(P_1)\gamma^\mu 2m\Lambda_+(P_3)\gamma^\nu\right)\\ &= \mathrm{Tr}\left(2m\Lambda_+(P_1)\gamma^\mu 2m\Lambda_+(P_3)\gamma^\nu\right), \end{aligned} \quad (4.84)$$

where we have used (1.58), (1.54), (1.65) and (1.114). Hence, the cross-section becomes[3]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{e^4}{64\pi s^2}\frac{1}{(P_3-P_1)^4}\mathrm{Tr}\left\{(\gamma\cdot P_3+m)\gamma^\mu(\gamma\cdot P_1+m)\gamma^\nu\right\}\times \\ \times\mathrm{Tr}\left\{(\gamma\cdot P_4+M)\gamma_\mu(\gamma\cdot P_2+M)\gamma_\nu\right\}, \quad (4.85)$$

where m is the electron mass and M is the muon mass. This result is the relativistic version of the Rutherford scattering formula. It agrees with the classical formula in the non-relativistic limit and also near the forward direction.

---

[3] It is possible to further simplify this expression using the gamma-matrix identities (proved in Tutorial Exercise 1):

$$\begin{aligned}&\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu},\\ &\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma}+g^{\nu\rho}g^{\mu\sigma}-g^{\mu\rho}g^{\nu\sigma}),\\ &\mathrm{Tr}\,(\text{odd no. of }\gamma\text{'s}) = 0,\end{aligned}$$

so that

$$\mathrm{Tr}\left\{(\gamma\cdot P_3+m)\gamma^\mu(\gamma\cdot P_1+m)\gamma^\nu\right\} = 4(m^2g^{\mu\nu}+P_1^\mu P_3^\nu+P_3^\mu P_1^\nu-P_1\cdot P_3 g^{\mu\nu}),$$

etc.

For e^-e^- scattering, there is also the second term in (4.82). In this case,

$$\begin{aligned}
|T|^2 &\to \tfrac{1}{4}\sum_{\text{spins}} T^*T \\
&= \frac{e^4}{4}\sum_{r,r',s,s'}\Bigg\{\frac{1}{(P_3-P_1)^4}\big(\bar u_r(P_3)\gamma^\mu u_s(P_1)\big)^*\big(\bar u_{r'}(P_4)\gamma_\mu u_{s'}(P_2)\big)^*\times \\
&\qquad\qquad \times\big(\bar u_r(P_3)\gamma^\nu u_s(P_1)\big)\big(\bar u_{r'}(P_4)\gamma_\nu u_{s'}(P_2)\big) \\
&\qquad -\frac{1}{(P_3-P_1)^2(P_4-P_1)^2}\big(\bar u_{r'}(P_4)\gamma^\mu u_s(P_1)\big)^*\big(\bar u_r(P_3)\gamma_\mu u_{s'}(P_2)\big)^*\times \\
&\qquad\qquad \times\big(\bar u_r(P_3)\gamma^\nu u_s(P_1)\big)\big(\bar u_{r'}(P_4)\gamma_\nu u_{s'}(P_2)\big) \\
&\qquad + (P_3\longleftrightarrow P_4)\Bigg\}. \qquad (4.86)
\end{aligned}$$

The first and third terms can be simplified using (4.84). To simplify the second and fourth terms, we have to use the fact that

$$\begin{aligned}
&\sum_{r,r',s,s'}\big(\bar u_{r'}(P_4)\gamma^\mu u_s(P_1)\big)^*\big(\bar u_r(P_3)\gamma_\mu u_{s'}(P_2)\big)^*\big(\bar u_r(P_3)\gamma^\nu u_s(P_1)\big)\big(\bar u_{r'}(P_4)\gamma_\nu u_{s'}(P_2)\big) \\
&= \sum_{r,r',s,s'}\big(\bar u_s(P_1)\gamma^\mu u_{r'}(P_4)\big)\big(\bar u_{s'}(P_2)\gamma_\mu u_r(P_3)\big)\big(\bar u_r(P_3)\gamma^\nu u_s(P_1)\big)\big(\bar u_{r'}(P_4)\gamma_\nu u_{s'}(P_2)\big) \\
&= \sum_{r,r',s,s'}\big(\bar u_s(P_1)\gamma^\mu u_{r'}(P_4)\big)\big(\bar u_{r'}(P_4)\gamma^\nu u_{s'}(P_2)\big)\big(\bar u_{s'}(P_2)\gamma_\mu u_r(P_3)\big)\big(\bar u_r(P_3)\gamma_\nu u_s(P_1)\big) \\
&= \sum_s \bar u_s(P_1)\gamma^\mu 2m\Lambda_+(P_4)\gamma^\nu 2m\Lambda_+(P_2)\gamma_\mu 2m\Lambda_+(P_3)\gamma_\nu u_s(P_1) \\
&= \sum_s \operatorname{Tr}\big(u_s(P_1)\bar u_s(P_1)\gamma^\mu 2m\Lambda_+(P_4)\gamma^\nu 2m\Lambda_+(P_2)\gamma_\mu 2m\Lambda_+(P_3)\gamma_\nu\big) \\
&= \operatorname{Tr}\big(2m\Lambda_+(P_1)\gamma^\mu 2m\Lambda_+(P_4)\gamma^\nu 2m\Lambda_+(P_2)\gamma_\mu 2m\Lambda_+(P_3)\gamma_\nu\big). \qquad (4.87)
\end{aligned}$$

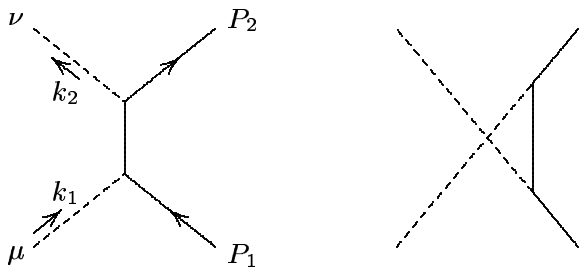
Hence, the cross-section becomes

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{e^4}{64\pi s^2}\Bigg\{&\frac{1}{(P_3-P_1)^4}\operatorname{Tr}\big\{(\gamma\cdot P_3+m)\gamma^\mu(\gamma\cdot P_1+m)\gamma^\nu\big\}\times \\
&\qquad\times\operatorname{Tr}\big\{(\gamma\cdot P_4+m)\gamma_\mu(\gamma\cdot P_2+m)\gamma_\nu\big\} \\
&-\frac{1}{(P_3-P_1)^2(P_4-P_1)^2}\operatorname{Tr}\big\{(\gamma\cdot P_1+m)\gamma^\mu(\gamma\cdot P_4+m)\gamma^\nu\times \\
&\qquad\qquad\times(\gamma\cdot P_2+m)\gamma_\mu(\gamma\cdot P_3+m)\gamma_\nu\big\} \\
&+(P_3\longleftrightarrow P_4)\Bigg\}, \qquad (4.88)
\end{aligned}$$

This result is the Mott cross-section.

4.7 Processes involving photons

We now turn to an example in QED involving photons, namely Compton scattering $\gamma e^- \to \gamma e^-$:



The Feynman rules can be used to deduce the amplitude:

$$T^{\mu\nu} = -e^2 \bar{u}(P_2)\left\{\gamma^\nu \frac{\gamma\cdot(k_1+P_1)+m}{(k_1+P_1)^2-m^2}\gamma^\mu + \gamma^\mu \frac{\gamma\cdot(P_1-k_2)+m}{(P_1-k_2)^2-m^2}\gamma^\nu\right\}u(P_1)\,. \tag{4.89}$$

The Lorentz indices in T label the initial and final polarisation state of the photon.

Suppose the spin states of the electron and the polarisation states of the photon are unknown. To calculate $|T|^2$, we handle the spinors in (4.89) as above. We also need to average over the (two) initial polarisation states of the photon and sum over the final ones:

$$\tfrac{1}{2}\sum_{\lambda,\kappa} |\varepsilon^\nu_{(\lambda)}(k_2)^* T_{\mu\nu}\varepsilon^\mu_{(\kappa)}(k_1)|^2, \tag{4.90}$$

where $\varepsilon_{(\lambda)}(k)$ are the polarisation vectors.

If $k^\mu = (k,0,0,k)$, a natural choice of polarisation vectors transverse to the direction of motion are (c.f. (3.123))

$$\varepsilon_{(1)}(k) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \varepsilon_{(2)}(k) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{4.91}$$

More generally, we have

$$\varepsilon_{(\lambda)}(k)\cdot\varepsilon_{(\kappa)}(k) = -\delta_{\lambda\kappa}\,, \qquad k_\mu \varepsilon^\mu_{(\lambda)}(k) = 0\,. \tag{4.92}$$

But these two relations are still not sufficient to fix $\varepsilon_{(\lambda)}(k)$: if we perform the transformation

$$\varepsilon^\mu_{(\lambda)}(k) \to \varepsilon^\mu_{(\lambda)}(k) + \Lambda_{(\lambda)}k^\mu\,, \tag{4.93}$$

for some constant $\Lambda_{(\lambda)}$, because $k^2 = 0$ for the photon, the new $\varepsilon_{(\lambda)}(k)$ also satisfies (4.92). One way to resolve this ambiguity is to introduce a fixed vector n^μ, and impose the further condition that

$$n_\mu e^\mu_{(\lambda)}(k) = 0\,. \tag{4.94}$$

For real $\varepsilon_{(\lambda)}(k)$, it can then be shown that

$$\sum_\lambda \varepsilon^\mu_{(\lambda)}(k)\varepsilon^\nu_{(\lambda)}(k) = -g^{\mu\nu} + \frac{n^\mu k^\nu + k^\mu n^\nu}{n\cdot k} - n^2\frac{k^\mu k^\nu}{(n\cdot k)^2}\,. \tag{4.95}$$

Furthermore, we have

$$k_\mu T^{\mu\nu} = 0 = T^{\mu\nu}k_\nu\,. \tag{4.96}$$

To prove this, note that

$$\begin{aligned}\gamma^\nu\frac{\gamma\cdot(k_1+P_1)+m}{(k_1+P_1)^2-m^2}\gamma\cdot k_1 &= \gamma^\nu\frac{\gamma\cdot(k_1+P_1)+m}{(k_1+P_1)^2-m^2}\big\{\gamma\cdot(k_1+P_1)-m-(\gamma\cdot P_1-m)\big\}\\ &= \gamma^\nu - \gamma^\nu\frac{\gamma\cdot(k_1+P_1)+m}{(k_1+P_1)^2-m^2}(\gamma\cdot P_1-m)\,.\end{aligned} \tag{4.97}$$

But the second term vanishes when multiplied into $u(P_1)$, by the Dirac equation. The remaining γ^ν is then cancelled by an identical contribution coming from the second term in (4.89). Thus, we arrive at (4.96). This is, in fact, a property of gauge invariance.

Using (4.96), it is readily seen that the second and third terms on the right-hand side of (4.95) do not contribute to the amplitude (4.90). In other words, the final result is independent of the choice of n^μ, as it should be. (4.95) effectively simplifies to $\frac{1}{2}(-1)^2 T^*_{\mu\nu}T^{\mu\nu}$. Hence, the amplitude is

$$\begin{aligned}|T|^2 &\to \tfrac{1}{2}\sum_{\text{spins}}\tfrac{1}{2}\sum_{\text{pol}}|\varepsilon^\nu_{(\lambda)}(k_2)^* T_{\mu\nu}\varepsilon^\mu_{(\kappa)}(k_1)|^2\\ &= \tfrac{1}{4}\sum_{\text{spins}} T^*_{\mu\nu}T^{\mu\nu},\end{aligned} \tag{4.98}$$

which can then be evaluated using the techniques of the previous section.

4.8 References and further reading

Good general references for this chapter include Mandl and Shaw, *Quantum Field Theory*, Peskin and Schroeder, *An Introduction to Quantum Field Theory*, and Roman, *Introduction to Quantum Field Theory*. Wick's theorem is proved in the second book. One of the

best treatments of ϕ^4 theory available is Ramond, *Field Theory: A Modern Primer*. A readable introduction to the formal theory of scattering can be found in Scadron, *Advanced Quantum Theory*.