8 Week 11

8.1 Gas giant planet formation

There are two basic models which have been proposed to explain the formation of gas giant planets. The *core accretion model* postulates that the envelopes of gas giants are accreted subsequent to the formation of a large core, which is itself assembled in a manner analogous to terrestrial planet formation (or more accurately through 'oligarchic growth'). Core accretion is the dominant theory for massive planet formation. The *gravitational instability* model, on the other hand, is based on the idea that some fraction of a massive protoplanetary disk might collapse directly to form massive planets. This has come under renewed theoretical scrutiny with the discovery of numerous extrasolar planets with masses much larger than that of Jupiter.

In this section, we review the physics of these theories. We also discuss the observational constraints on the different theories, which include inferences as to the core masses of the gas giants in the Solar System, the host metallicity/planet frequency correlation for extrasolar planetary systems, and — indirectly — comparison of the theoretically derived time scales with observations of protoplanetary disk lifetimes. This is a critical issue, since gas giants must form prior to the dispersal of the gas disk. Any successful model of massive planet formation must grow such bodies within at most 5-10 Myr.

8.1.1 Core accretion model

The main stages in the formation of a gas giant via core accretion are illustrated schematically in Figure 1. A core of rock and ice forms via the same mechanisms that we have previously outlined for terrestrial planet formation. Initially, there is either no atmosphere at all (because the gravitational potential is too shallow to hold on to a bound atmosphere), or any gas is dynamically insignificant. However, as the core grows, eventually it becomes massive enough to hold on to a significant envelope. At first, the envelope is able to maintain hydrostatic equilibrium. The core continues to grow via accretion of planetesimals, and the gravitational potential energy liberated as these planetesimals rain down on the core provides the main source of luminosity. This growth continues until the core reaches a *critical mass*. Once the critical mass is reached, the envelope can no longer be maintained in hydrostatic equilibrium. The envelope contracts on its own Kelvin-Helmholtz time scale (which can be very long - millions of years), and eventually a phase of rapid gas accretion occurs. This process continues until (a) the planet becomes massive enough to open up a gap in the protoplanetary disk, thereby slowing down the rate of gas supply, or (b) the gas disk itself is dispersed.

The novel aspect of the core accretion model is the existence of a critical core mass. Numerical models have been computed which demonstrate the existence of a maximum core mass, and show that it depends only weakly on the local properties of the *gas* within the protoplanetary disk. Here, we show that a toy model in which energy transport is due solely to radiative diffusion displays the key property of a critical core mass.

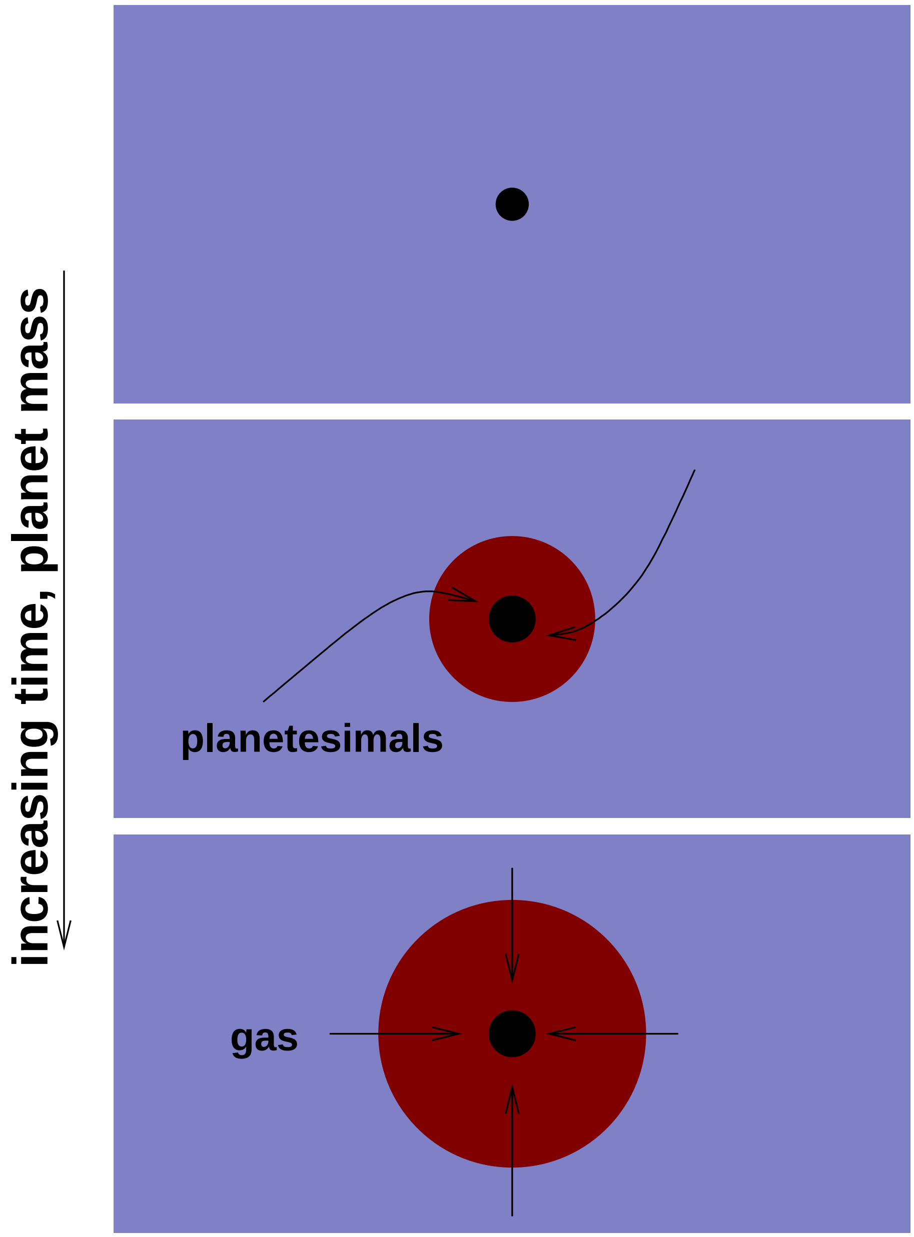


Figure 1: Illustration of the main stages of the core accretion model for giant planet formation.

Consider a core of mass $M_{\rm core}$ and radius $R_{\rm core}$, surrounded by a gaseous envelope of mass $M_{\rm env}$. The total mass of the planet,

$$M_t = M_{\rm core} + M_{\rm env}. \tag{162}$$

The envelope extends from $R_{\rm core}$ to some outer radius $R_{\rm out}$, which marks the boundary between the gas bound to the planet and the gas in the protoplanetary disk. $R_{\rm out}$ may be determined by thermal effects (in which case $R_{\rm out} \sim GM_t/c_s^2$, with c_s the disk sound speed – determined essentially from considering the escape velocity of gas particles from the planet) or by tidal considerations (giving an outer radius of R_{Hill}), whichever is the smaller. If the envelope is of low mass, then the largest contribution to the luminosity is from accretion of planetesimals onto the core. This yields a luminosity,

$$L = \frac{GM_{\rm core}\dot{M}_{\rm core}}{R_{\rm core}} \tag{163}$$

which is constant through the envelope.

If we assume that radiative diffusion dominates the energy transport, then the structure of the envelope is determined by the equations of hydrostatic equilibrium and radiative diffusion,

$$\frac{{\rm d}P}{{\rm d}r} = -\frac{GM(r)}{r^2}\rho \tag{164}$$

$$\frac{L}{4\pi r^2} = -\frac{16}{3}\frac{\sigma T^3}{\kappa_R \rho}\frac{{\rm d}T}{{\rm d}r} \tag{165}$$

where σ is the Stefan-Boltzmann constant and κ_R the Rosseland mean opacity (here assumed constant). Adopting an ideal gas equation of state,

$$P = \frac{\mathcal{R}}{\mu}\rho T, \tag{166}$$

where $\mathcal{R}$ is the gas constant and μ the mean molecular weight, and approximating the derivatives in the above equations as ratios — i.e. ${\rm d}P/{\rm d}r \sim P/r$ — we obtain,

$$T \simeq \left(\frac{\mu}{\mathcal{R}}\right)\frac{GM_t}{r} \tag{167}$$

$$\rho \simeq \frac{64\pi\sigma}{3\kappa_R L}\left(\frac{\mu}{\mathcal{R}}GM_t\right)^4\frac{1}{r^3} \tag{168}$$

In deriving these expressions we have additionally assumed that $M(r) \simeq M_t$, which is reasonable if the envelope mass is not too large. Integrating the density profile we obtain the envelope mass,

$$\begin{aligned} M_{\rm env} &= \int_{R_{\rm core}}^{R_{\rm out}} 4\pi r^2 \rho(r) dr \\ &= \frac{256\pi^2\sigma}{3\kappa_R L}\left(\frac{\mu}{\mathcal{R}}GM_t\right)^4 \ln\left(\frac{R_{\rm out}}{R_{\rm core}}\right). \end{aligned} \tag{169}$$

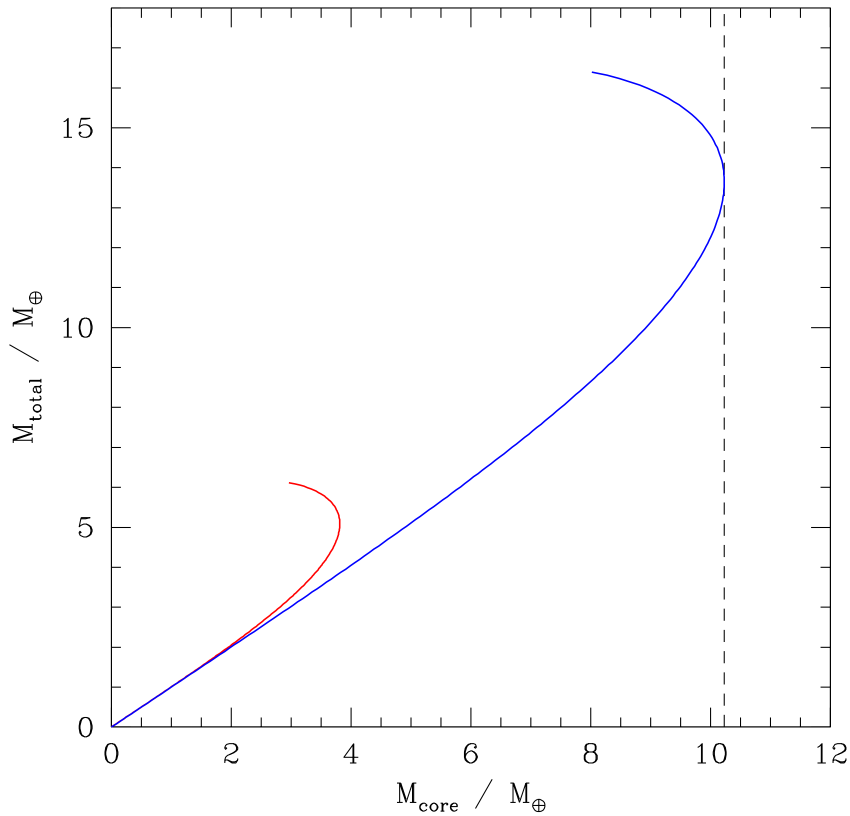


Figure 2: Solutions to equations (171) for the core mass $M_{\rm core}$ and total mass $M_{\rm total}$. The blue curve is for a higher planetesimal accretion rate than for the red curve. The critical core mass is shown as the vertical dashed line. One should not take solutions to this toy model very seriously, but the numbers have been fixed here to correspond roughly to the values obtained from real calculations.

Noting that for a uniform density core,

$$L \propto \frac{M_{\rm core}\dot{M}_{\rm core}}{R_{\rm core}} \propto M_{\rm core}^{2/3}\dot{M}_{\rm core}, \tag{170}$$

and approximating the logarithmic terms as constants, we obtain finally,

$$\begin{aligned} M_t &= M_{\rm core} + \frac{K}{\kappa_R}\frac{M_t^4}{M_{\rm core}^{2/3}\dot{M}_{\rm core}} \\ M_{\rm core} &= M_t - \frac{K}{\kappa_R}\frac{M_t^4}{M_{\rm core}^{2/3}\dot{M}_{\rm core}}. \end{aligned} \tag{171}$$

Here K absorbs the numerous 'constant' terms in equation (169).

Solutions to equation (171), which govern how the total planet mass depends on the core mass, are plotted in figure 2. One sees that for fixed $\dot{M}_{\rm core}$, there exists a maximum or critical core mass $M_{\rm crit}$ beyond which no solution is possible. The physical interpretation of this result is that if one tries to build a planet with a core mass above the critical mass hydrostatic equilibrium cannot be achieved in the envelope. Rather the envelope will contract, and further gas will fall in as fast as gravitational potential energy can be radiated. This occurs because the luminosity provided by planetesimal accretion provides too little thermal energy to the envelope to support it against the gravity of the core.

This toy model should not be taken too seriously, but it does illustrate the most important result from more detailed calculations — namely that the critical mass increases with larger $\dot{M}_{\rm core}$ and with enhanced opacity. An approximate fit to published results from computer simulations is given by,

$$\frac{M_{\rm crit}}{M_\oplus} \approx 12 \left(\frac{\dot{M}_{\rm core}}{10^{-6} \ M_\oplus {\rm yr}^{-1}} \right)^{1/4} \left(\frac{\kappa_R}{1 \ {\rm cm}^2 {\rm g}^{-1}} \right)^{1/4} \tag{172}$$

where the power-law indices are uncertain by around ± 0.05. The weak dependence of the critical core mass on the planetesimal accretion rate means that, within a particular core accretion model, we can always speed up the approach to unstable gas accretion simply by increasing the assumed surface density of planetesimals in the vicinity of the growing core. But one still has to wait for a Kelvin-Helmholtz time for the envelope to contract so that more gas can accrete onto the planet whose outer radius during early phases is equal to the Hill sphere radius.

Existing calculations of giant planet formation via core accretion are very detailed, and show that in general it required several million years to build a Jupiter (see the accompanying power point slide). However, there are a number of uncertainties in these models which provide considerable leaway in these time scale estimates:

- **What is the magnitude of the opacity?** Although κ_R enters equation (172) as rather a weak power, its magnitude is highly uncertain. A couple of research groups have recently computed new core accretion models in which the opacity is reduced from the interstellar value by a factor of 50. This allows for much more rapid formation of gas giants than was obtained in prior models. What matters most is the opacity in the upper regions of the envelope. There is some motivation for considering smaller values of the opacity due to grain growth and settling in the planet atmosphere – but it remains unclear how accurately we can determine what the appropriate value to use is.

- **The neglect of Type I migration of growing cores**. Theoretical work, which we will discuss more fully in a subsequent section, suggests that planets or planetary cores with masses exceeding $1 M_\oplus$ are highly vulnerable to radial migration as a consequence of gravitational torques exerted by the gas disk. This effect is not included in most calculations of growing planets, but the few research groups to have examined this effect indicate that it can change formation time scales considerably. In part this is because a larger core can be formed – because it does not exhaust its feeding zone, resulting in the more rapid formation of a gas giant planet.

To summarize, the broad outlines of how core accretion works are well established, but further work is needed to delineate under what circumstances (i.e. for what values of the surface density, disk lifetime, migration rates and envelope opacity) it results in successful formation of a massive planet.

8.1.2 Gravitational instability model

A sufficiently massive and/or cold gas disk is gravitationally unstable[1]. If — and this is a big if — gravitational instability leads to *fragmentation* this can lead to massive planet formation.

The necessary condition for gravitational instability to occur is that the Toomre Q parameter be low enough, specifically,

$$Q \equiv \frac{c_s \Omega}{\pi G \Sigma} < Q_{\rm crit} \simeq 1 \tag{173}$$

where c_s is the sound speed in a gas disk of local surface density Σ. This criterion comes from considering the gravitational stability of a self-gravitating disc using a linear perturbation analysis. If we consider a disk with $h/r = 0.05$ at 10 AU around a Solar mass star, then the relation $h/r = c_s/v_\phi$ yields a sound speed $c_s \simeq 0.46$ kms^{-1}. To attain $Q = 1$, we then require a surface density,

$$\Sigma \approx 1.4 \times 10^3 \ {\rm g\,cm}^2. \tag{174}$$

This is much larger than estimates based, for example, on the minimum mass Solar Nebula, from which we conclude robustly that gravitational instability is most likely to occur at an early epoch when the disk mass is still high. The characteristic wavelength for gravitational instability is given by $\lambda_{\rm crit} = 2c_s^2/(G\Sigma)$ (which is also obtained from consideration of the gravitational stability of a disc using linear perturbation analysis), so we find that the mass of objects formed if such a disk fragmented would be,

$$M_p \sim \pi \Sigma \lambda_{\rm crit}^2 \sim \frac{4\pi c_s^4}{G^2 \Sigma} \sim 3 M_J \tag{175}$$

where M_J is the mass of Jupiter. These order of magnitude estimates suffice to indicate that gravitational instability followed by fragmentation could form gas giants.

It is also straightforward to derive where in the disk gravitational instability is most likely to occur. Noting that in a steady-state accretion disk $\nu\Sigma = \dot{M}/(3\pi)$, we use the α prescription $\nu = \alpha c_s^2/\Omega$ and obtain,

$$Q \propto \frac{c_s^3}{\dot{M}}. \tag{176}$$

The sound speed in a protoplanetary disk decreases outward, so a steady-state disk becomes less stable at large radii. Indeed, unless the temperature becomes so low that

[1]The terminology used to discuss this process is potentially confusing. We will use the term *gravitational instability* to refer to disks in which the self-gravity of the gas is significant enough to alter the structure or evolution of the disk. *Fragmentation* refers to the case where gravitational instability leads to the breakup of the disk into bound objects.

external irradiation (not that from the central star) dominates the heating, a steady-state disk *will* become gravitational unstable provided only that it is large enough.

To derive sufficient conditions for fragmentation, we need to go beyond these elementary considerations and ask what happens to a massive disk as instability is approached. The critical point is that as Q is reduced, *non-axisymmetric* instabilities set in which do not necessarily lead to fragmentation. Rather, the instabilities generate spiral arms which both transport angular momentum and lead to dissipation and heating. The dissipation in particular results in heating of the disk, which raises the sound speed and makes fragmentation less likely. On a longer time scale, angular momentum transport also leads to lower surface density and, again, enhanced stability.

Given these consideration, when will a disk fragment? Both analytic arguments and local numerical simulations have been used to identify the *cooling time* as the control parameter determining whether a gravitationally unstable disk will fragment. For an annulus of the disk we can define the equivalent of the Kelvin-Helmholtz time scale for a star,

$$t_{\rm cool} = \frac{U}{2\sigma T_{\rm disk}^4} \tag{177}$$

where U is the thermal energy content of the disk per unit surface area. Then for an ideal gas equation of state with $\gamma = 5/3$ the boundary for fragmentation is:

- $t_{\rm cool} \lesssim 3\Omega^{-1}$ — the disk fragments.
- $t_{\rm cool} \gtrsim 3\Omega^{-1}$ — disk reaches a steady state in which heating due to dissipation of gravitational turbulence balances cooling.

This condition is intuitively reasonable. Spiral arms resulting from disk self-gravity compress patches of gas within the disk on a time scale that is to order of magnitude Ω^{-1}. If cooling occurs on a time scale that is shorter that Ω^{-1}, the heating due to adiabatic compression can be radiated away, and in the absence of extra pressure collapse is likely.

None of the above is the subject of much theoretical doubt. Whether a massive protoplanetary disk can fragment into massive planets depends upon its cooling time. What remains controversial is whether the cooling time scale can, in fact, ever be short enough. Analytic arguments suggest that attaining a short enough cooling time scale is difficult, especially at small orbital radii, and that the most likely scenario for fragmentation would involve very massive planets (or brown dwarfs) forming at radii of the order of 50 or 100 AU. Simulations yield a contradictory picture at the present time, with different research groups arguing for or against the possibility of gravitational instability. Clearly there is more work to do in this area !

8.1.3 Comparison with observations

The architecture of the Solar System's giant planets provides qualified support for the core accretion model. The time scale for core accretion increases with orbital radius, which is qualitatively consistent with the general trend of planetary properties in the outer Solar System — Jupiter is closest to Solar composition (reflecting a fully formed gas giant), while Saturn and the ice giants are relatively gas poor. Perhaps these outermost planets formed as the gas disk was in the process of being dispersed. Explaining the origin of Uranus and Neptune as a consequence of disk fragmentation is not easy. Moreover the core accretion time scale for the formation of Jupiter — about 8 Myr in the most detailed calculation — is reasonable for plausible assumptions. Applying the model to extrasolar planetary systems, we would expect that a greater surface density of planetesimals would lead to faster core growth and an increased chance of reaching runaway gas accretion prior to disk dispersal. This is consistent with the observed correlation of planet frequency with host star metallicity. It is currently unclear whether this correlation — which appears to reflect the formation process – could also be consistent with disk fragmentation.

Solar System observations also raise doubts about core accretion. The time scale to form Neptune, in particular, is prohibitively long. This result is now normally interpreted as an indication that Uranus and Neptune may not have formed in situ, and as such cannot be used to argue against core accretion. It means, however, that the ice giants are poor laboratories for testing core accretion. Potentially more seriously, a combination of *Galileo* measurements and interior structure models places strong constraints on the maximum core mass of Jupiter. Some models obtain an upper limit on the core mass of Jupiter of $10 M_\oplus$ for the most optimistic choice of equation of state (optimistic in the sense of yielding the weakest constraints). For some equations of state the constraint on the core mass can be below $5 M_\oplus$[2]. This is smaller than predictions based on the simplest models of core accretion, and is completely consistent with the zero core prediction of disk instability. However as we have already noted fiducial core accretion models are based on particular choices of uncertain parameters and as such should not be regarded as definitive. Currently, it seems reasonable to believe that smaller core masses — perhaps as low as $5 M_\oplus$ — could be consistent with plausible variants of the basic core accretion model. Of course if refinements to the high pressure equation of state lead to the conclusion that Jupiter is genuinely devoid of a core, then that would spell serious trouble for core accretion. Similarly the discovery of massive planets at very large orbital radii — where disk instability is most likely and the time scale for core accretion very large — appear to support fragmentation models, though it may be hard to rule out the possibility that any such planets formed closer to the star and migrated outward.

Although we have phrased this discussion in terms of either core accretion or disk fragmentation providing a mechanism for massive planet formation, it of course remains possible that both could be viable formation channels. If so, the most likely scenario would see core accretion forming lower mass planets at small orbital radii, while gravitational instability would yield very massive planets typically further out. The existence of two channels could be inferred, for example, by looking for *different* metallicity distributions of stars hosting high and low mass planets.

[2] The same exercise yields a core mass for Saturn of 10-20$M_\oplus$, in good accord with the expectations of core accretion

8.2 Planetary migration

The story is not over once planets have managed to form. Theoretical models, which are now strongly supported by observations of the Solar System and of extrasolar planetary systems, suggest at least three mechanisms that can lead to substantial post-formation orbital evolution:

- **Interaction between planets and the gaseous protoplanetary disk**. This leads to orbital *migration* as a consequence of angular momentum exchange between the planet and the gas disk, and can be important for both terrestrial-mass planets and gas giants while the gas disk is still present. Gas disk migration provides the standard theoretical explanation for the existence of hot Jupiters.

- **Interaction between planets and a remnant planetesimal disk**. Planets, especially giant planets, can also exchange angular momentum by interacting with and ejecting planetesimals left over from the planet formation process. This mechanism appears likely to have caused orbital migration of at least the ice giants, and possibly also Saturn, during the early history of the Solar System.

- **Interaction within an initially unstable system of two or more massive planets**. There is no guarantee that the architecture of a newly formed planetary system will be stable over the long run. Instabilities can lead to planet-planet scattering, which usually results in the ejection of the lower mass planets, leaving the survivors on eccentric orbits. This could be the origin of the typically eccentric orbits seen in extrasolar planetary systems.

In this section we discuss the first of these mechanisms, and we will discuss the second mechanism next week.

8.3 Gas disk migration

The most detailed calculations of the rate of angular momentum exchange between a planet and a gas disk are based on summing the torques exerted at discrete *resonances* within the disk. This calculation is too lengthy and technical to reproduce here. Here we summarize the conditions for resonances to exist, and discuss the effect of the torques on the planet and on the disk in the limits of high and low planet masses.

8.3.1 Conditions for resonance

We consider a planet orbiting a star on a circular orbit with angular frequency Ω_p. A standard exercise in dynamics (e.g. Binney & Tremaine 1987) yields the conditions for resonances. A *corotation resonance* exists for radii in the disk where the angular frequency Ω,

$$\Omega = \Omega_p. \tag{178}$$

Lindblad resonances exist when,

$$m(\Omega - \Omega_p) = \pm\kappa_0 \tag{179}$$

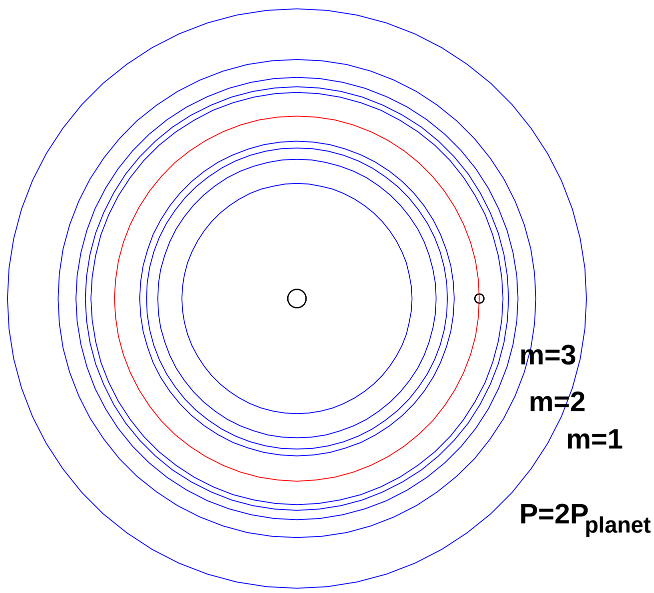


Figure 3: Nominal locations of the corotation (red) and Lindblad resonances (blue) for a planet on a circular orbit. Only the low order Lindblad resonances are depicted — there are many more closer to the planet.

where m is an integer and κ_0, the *epicyclic frequency*, is defined as,

$$\kappa_0 \equiv \left(\frac{\mathrm{d}^2 \Phi_0}{\mathrm{d}r^2} + 3\Omega^2 \right) \tag{180}$$

with Φ_0 the stellar gravitational potential. The epicyclic frequency, κ_0, is the natural frequency of radial oscillation for an orbiting particle, and the Lindblad resonances occur at locations where the forcing frequency (as seen by the perturbed particle) is equal to the natural oscillation frequency. For a Keplerian potential $\kappa_0 = \Omega$. If we approximate the angular velocity of gas in the disk by the Keplerian angular velocity, the Lindlad resonances are located at,

$$r_L = \left(1 \pm \frac{1}{m} \right)^{2/3} r_p \tag{181}$$

where r_p is the planet orbital radius. The locations of some of the low order (small m) resonances are shown in Figure 3. For an orbiting test particle, the resonances are locations where the planet can be a strong perturbation to the motion. For a gas disk, angular momentum exchange between the planet and the gas disk occurs at resonant locations.

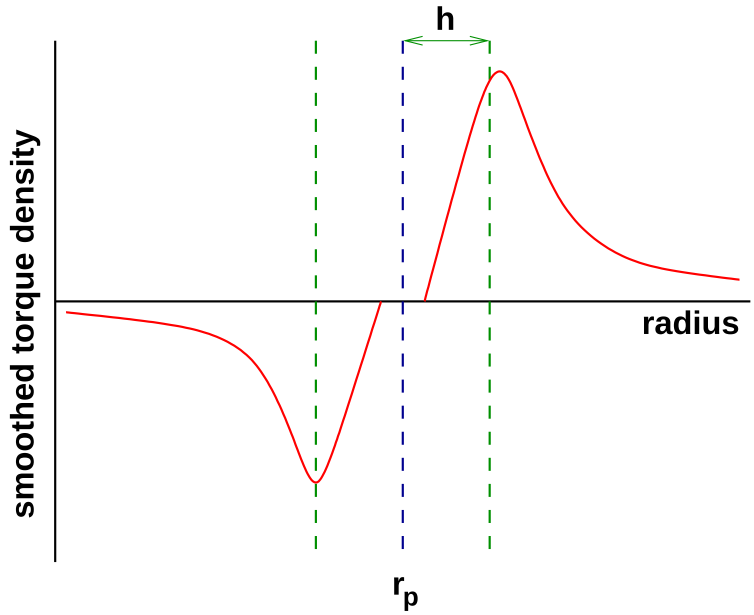


Figure 4: Schematic illustration of the smoothed torque density due to angular momentum exchange between a planet and a gas disk at the location of Lindblad resonances, after [**?**]. The peak torque occurs at $r \approx r_p \pm h$. The disk gains angular momentum from the planet as a result of the interaction for $r > r_p$, and loses angular momentum for $r < r_p$. The interaction is almost invariably asymmetric, such that when integrated over the entire disk the planet loses angular momentum and migrates inward.

8.3.2 Gravitational torques at resonances

For a planet on a circular orbit embedded within a geometrically thin protoplanetary disk, angular momentum exchange between the planet and the gas occurs at the location of the resonances defined by equation (178) and (179). The sense of the exchange is that,

- The planet **gains angular momentum** from interacting with the gas disk at the interior Lindblad resonances ($r_L < r_p$). This tends to drive the planet outward. The gas loses angular momentum, and moves inward.
- The planet **loses angular momentum** from interacting with the gas disk at exterior Lindblad resonances ($r_L > r_p$). This tends to drive the planet toward the star. The gas gains angular momentum, and moves outward.

Notice that the gravitational interaction of a planet with a gas disk tends — somewhat counter-intuitively — to *repel* gas from the vicinity of the planet's orbit.

The flux of angular momentum exchanged at each Lindblad resonance can be written as,

$$T_{LR}(m) \propto \Sigma M_p^2 f_c(\xi) \tag{182}$$

where Σ is the gas density and M_p the planet mass. That the torque should scale with the *square* of the planet mass is intuitively reasonable — the perturbation to the disk surface density scales as the planet mass in the linear regime so the torque scales as M_p^2. The factor $f_c(\xi)$ is the *torque cutoff function*, which encodes the fact that resonances very close to the planet contribute little to the net torque, and ξ is defined by $r = r_p \pm \xi$. The torque cutoff function peaks at,

$$\xi \equiv m \left(\frac{c_s}{r\Omega} \right)_p \simeq 1 \tag{183}$$

i.e. at a radial location $r \simeq r_p \pm h$, where h is the disk scale height (this result immediately implies that a three-dimensional treatment is necessary for the dominant resonances if the planet is completely embedded within a gas disk, as is the case for low mass planets). The strength of the torque exerted at each resonance is essentially independent of properties of the disk such as the disk viscosity, though of course the viscosity still matters inasmuch as it controls the value of the unperturbed disk surface density Σ.

Figure 4 illustrates the differential torque exerted on the disk by the planet, after smoothing over the Lindblad resonances. The flux of angular momentum is initially deposited in the disk as waves, which propagate radially before dissipating. The details of this dissipation matter little for the net rate of angular momentum exchange.

Angular momentum transfer at the corotation resonance requires additional consideration. In a two-dimensional disk, the rate of angular momentum deposition at corotation is proportional

$$T_{CR} \propto \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\Sigma}{B} \right) \tag{184}$$

where B is the Oort parameter,

$$B(r) = \Omega + \frac{r}{2} \frac{\mathrm{d}\Omega}{\mathrm{d}r}. \tag{185}$$

This implies that in a two-dimensional disk, the corotation torque vanishes identically in the moderately interesting case of a disk with a surface density profile $\Sigma \propto r^{-3/2}$.

8.3.3 Type I migration

For low mass planets (generically $M_p \sim M_\oplus$, though the exact mass depends upon the disk properties) the angular momentum flux injected into the disk as a consequence of the planet-disk interaction is negligible when compared to the viscous transport of angular momentum. As a result, the gas surface density profile $\Sigma(r)$ remains approximately unperturbed, gas is present at the location of each of the resonances, and the net torque on the planet is obtained by summing up the torque exerted at each resonance. Schematically,

$$T_{\text{planet}} = \sum_{ILR} T_{LR} + \sum_{OLR} T_{LR} + T_{CR} \tag{186}$$

where the planet gains angular momentum from the inner Lindblad resonances (ILR) and loses angular momentum to the outer Lindblad resonances (OLR). Changes to the planet's orbit as a result of this net torque are called **Type I migration**.

As noted above (equation 182) the torque exerted at each resonance scales as the planet mass squared. If the azimuthally averaged surface density profile of the gas disk remains unperturbed, it follows that the total torque will also scale as M_p^2 and the migration time scale,

$$\tau_I = \frac{J_p}{T_{\rm planet}} = \frac{M_p R_p^2 \Omega_p}{T_{\rm planet}} \propto M_p^{-1}. \tag{187}$$

Type I migration is therefore most rapid for the largest body for which the assumption that the gas disk remains unaffected by the planet remains valid.

Invariably it is found that the Lindblad resonances exterior to the planet are more powerful than those interior (in large part because they lie closer to the planet due to partial pressure support in the disk causing the disc to rotate with a sub-Keplerian angular velocity), so that the net torque due to Lindblad resonances leads to *inward* migration. The torque at corotation is of opposite sign and of comparable magnitude to the net Lindblad torque, but is not usually strong enough to reverse the sense of migration. Note that one might think (for example by looking at the surface density dependence of the torque in equation 182) that the sense of migration ought to depend upon the surface density gradient — i.e. that a steep surface density profile should strengthen the inner resonances relative to the outer ones and hence drive outward migration. This *is not true*. Pressure gradients (which depend upon the radial dependence of the surface density and temperature) alter the angular velocity in the disk from its Keplerian value, and thereby shift the radial location of resonances from their nominal positions. A steep surface density profile implies a large pressure gradient, so that all the resonances move slightly inward. This weakens the inner Lindblad resonance relative to the outer ones, in such a way that the final dependence on the surface density profile is surprisingly weak.

For a 3D isothermal disk in which,

$$\Sigma(r) \propto r^{-\gamma} \tag{188}$$

the migration time scale is given by,

$$\begin{aligned} \tau_I &\equiv -\frac{r_p}{\dot{r}_p} \\ &= (2.7 + 1.1\gamma)^{-1} \frac{M_*}{M_p} \frac{M_*}{\Sigma_p r_p^2} \left(\frac{c_s}{r_p \Omega_p} \right)^2 \Omega_p^{-1}, \end{aligned} \tag{189}$$

where Σ_p, c_s and Ω_p are respectively the gas surface density, gas sound speed, and angular velocity at the location of a planet orbiting at distance r_p from a star of mass M_*. As expected based on the simple considerations discussed previously, the migration rate ($\propto \tau_I^{-1}$) scales linearly with both the planet mass and the local disk mass. The time scale becomes shorter for cooler, thinner disks — provided that the interaction remains in the Type I regime — since for such disks more resonances close to the planet contribute to the net torque.

Figure 5 shows the predicted migration time scale as a function of radius for a $5 M_\oplus$ core in a disk with $h/r = 0.05$ and $\Sigma \propto r^{-1}$. Two disk masses are plotted,

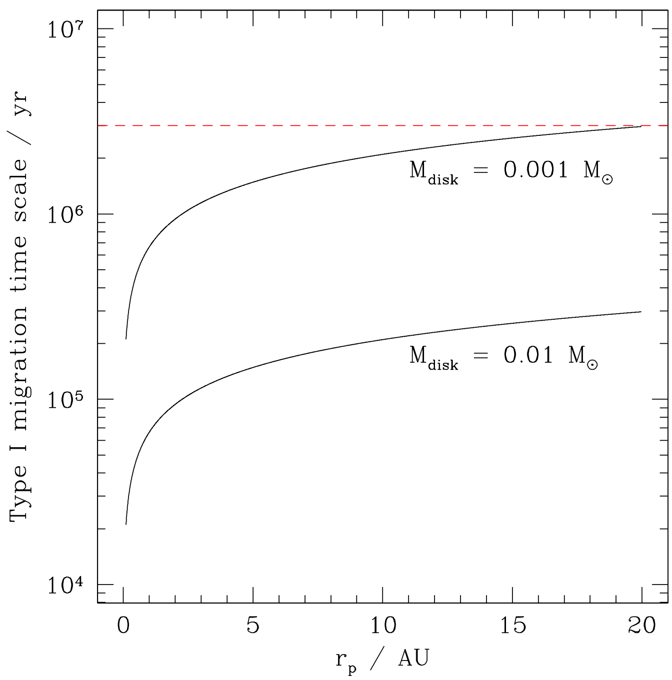


Figure 5: The inward Type I migration time scale for a $5M_\oplus$ core as a function of orbital radius, calculated using the three-dimensional isothermal disk formula of [**?**]. The lower curve assumes a disk with $\Sigma \propto r^{-1}$, $h/r = 0.05$, and a total mass of $0.01M_\odot$ within 30 AU. The upper curve shows the migration time scale in a similar disk with a mass of only $0.001M_\oplus$ — the absolute minimum needed to form a Jupiter mass planet. The red dashed line illustrates a typical estimate for the lifetime of the gas disk.

one in which the disk mass interior to 30 AU is $10^{-2}M_{\odot}$, and one in which the disk mass is $10^{-3}M_{\odot}$. As is obvious from the figure, the migration time scale from radii close to the snow line is a small fraction of the disk lifetime for the more massive disk model. One concludes that — unless the torque calculation is missing essential physics that qualitatively changes the answer — Type I migration is likely to be an essential element of giant planet formation via core accretion. Only if the disk mass is very low (almost the absolute minimum needed to form a gas giant at all) can the effects of Type I migration be reduced. It may be that achieving successful planet formation via core accretion requires waiting until the gas disk is weak enough to slow Type I migration to a manageable rate.

8.4 Gap formation by a giant planet and type II migration

The discussion presented in the previous section shows that solid planetary cores are able to form within the life time of a protoplanetary accretion disc. Detailed calculations indicate that a solid core containing ~ 15 M_{Earth} needs to form before a gas giant planet (e.g. Jupiter) can form by the accretion of gas onto the solid core. For core masses lower than this the disc gas remains hydrostatically supported, and a gas giant planet is unable to form. In order to form such a massive core within the life time of protostellar discs, it must form beyond a radius of $\sim 3-5$ AU where the temperature in the nebula is such that ices can begin to form. These ices augment the amount of solid material available, and may increase the ability of solids to stick on impact. Once a ~ 15 M_{Earth} core forms, the gas accretion process requires between $10^6 - 10^7$ yr.

High mass planets interact tidally with the accretion disc, and can form gaps within them. Angular momentum exchange between the planet and disc can cause orbital migration, but the basic picture for high mass planets differs from that just discussed for low-mass, non gap forming planets.

8.4.1 Gap Formation Criterion

We will use a simple impulse approximation to calculate the angular momentum exchange between a disc and a planet.

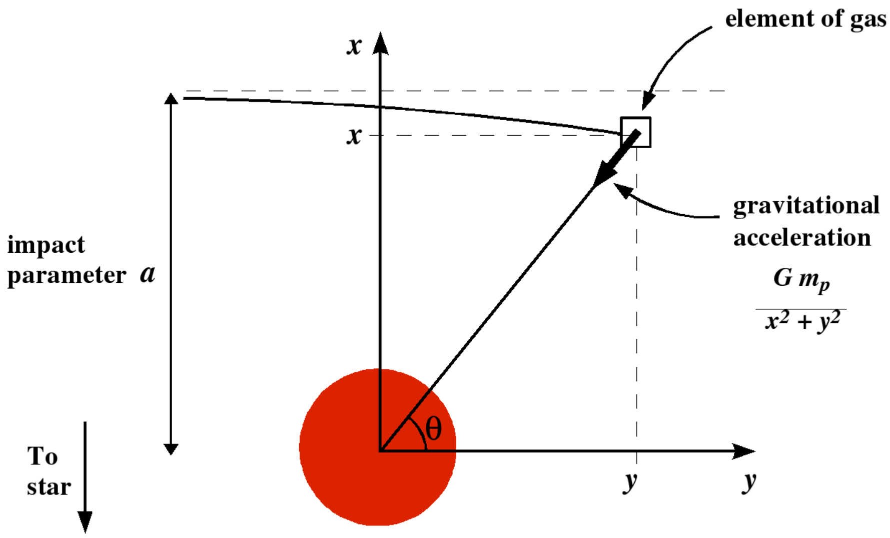
We will work in a reference frame rotating with the planet at radius R. The material exterior to the planet orbital radius streams past with an unperturbed speed $u = -R\,\Omega'\,x$, where $\Omega' = \mathrm{d}\Omega/\mathrm{d}R$. Let the impact parameter be a. The material suffers a deflection as it streams past the planet.

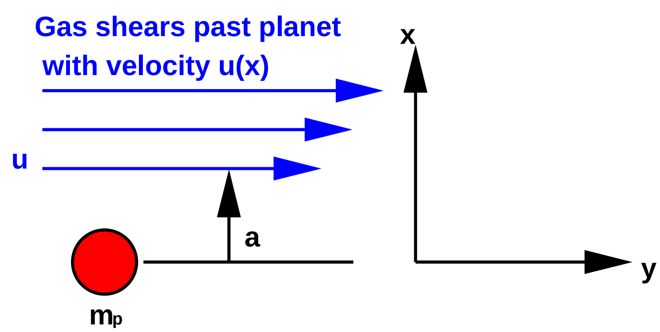
The equation of motion of the material that streams past the planet is

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{Gm_p x}{(x^2+y^2)^{3/2}} \ . \tag{190}$$

We now set $x = a$ and $y = ut$, so that equation (190) becomes

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{G\,m_p\,a}{(a^2+u^2t^2)^{3/2}} \ . \tag{191}$$





The change in v_x during the encounter is then

$$\Delta v_x \;=\; \int_{-\infty}^{\infty} -\frac{G\,m_p\,a\,\mathrm{d}t}{(a^2+u^2t^2)^{3/2}} \;. \tag{192}$$

Making the substitution $ut = a\tan\theta$ we obtain

$$\Delta v_x \;=\; -\int_{-\pi/2}^{\pi/2} \frac{G\,m_p\,a^2\sec^2\theta\mathrm{d}\theta}{ua^3\sec^3\theta} \;=\; -\frac{2Gm_p}{ua} \;. \tag{193}$$

Thus,

$$(\Delta v_x)^2 \;=\; \frac{4(G\,m_p)^2}{u^2a^2}.$$

But the total kinetic energy change due to the encounter $= 0$, so

$$|v_y+\Delta v_y|^2 \;+\; (\Delta v_x)^2 \;=\; v_y^2$$

and assuming that $\Delta v_y \ll v_y$ we find that

$$\Delta v_y \;=\; -\frac{(\Delta v_x)^2}{2u} \;=\; -\frac{2(Gm_p)^2}{u^3a^2} \;. \tag{194}$$

The angular momentum exchanged is $R\Delta v_y$ per unit mass. The rate of angular momentum exchange, $\dot{J}$ is given by the amount of angular momentum exchange per encounter divided by the time between each encounter. The time between each encounter is given by

$$\frac{2\pi R}{|\Omega' a R|}$$

Thus we can write an equation for $\dot{J}$ as follows

$$\dot{J} \;=\; \int_{a_{min}}^{\infty} \left(\frac{2R(Gm_p)^2}{u^3 a^2} \right) \frac{2\pi R \Sigma \,\mathrm{d}a}{\left(\frac{2\pi R}{|\Omega' a R|}\right)} \; . \tag{195}$$

Setting $u = |\Omega' R a|$ we get

$$\begin{aligned} \dot{J} \;&=\; \int_{a_{min}}^{\infty} \frac{2(Gm_p)^2 R^3 \Sigma a |\Omega'|}{\Omega'^3 R^4 a^5} \,\mathrm{d}a \\ &=\; \int_{a_{min}}^{\infty} \frac{2(Gm_p)^2 \Sigma}{\left(\frac{3\Omega}{2R}\right)^2 a^4 R} \,\mathrm{d}a \; . \end{aligned} \tag{196}$$

Integrating gives the rate at which angular momentum is given to the outer disc material by the planet

$$\begin{aligned} \dot{J} \;&=\; \frac{8(Gm_p)^2 \Sigma R^2}{27 \Omega^2 a_{min}^3 R} \\ &=\; \frac{8}{27} \left(\frac{m_p}{M_\odot} \right)^2 \Sigma R^4 \left(\frac{R}{a_{min}} \right)^3 \Omega^2 \end{aligned} \tag{197}$$

where we have assumed that the central star is one solar mass. Note that the angular momentum exchange is smaller for a wider gap.

For a massive gas giant planet we set a_{min} = planet Hill sphere radius = $R(q/3)^{1/3}$ where $q = m_p/M_\odot$ assuming the central star is one solar mass. This gives

$$\begin{aligned} \dot{J} \;&=\; \frac{24}{27} \left(\frac{m_p}{M_\odot} \right)^2 \frac{\Sigma R^2 M_\odot}{m_p} R^2 \Omega^2 \\ &=\; \left(\frac{24}{27} \right) \frac{m_p}{M_\odot} (\Sigma R^2)(R^2 \Omega^2) \; . \end{aligned} \tag{198}$$

The condition for a gap to form is that the tidal torques acting on the disc due to the planet be greater than the internal viscous torques acting in the disc. Thus we obtain

$$\left(\frac{24}{27} \right) \frac{m_p}{M_\odot} (\Sigma R^2)(R^2 \Omega^2) \;>\; -2\pi R^3 \Sigma \nu \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R} \right) \;=\; 3\pi R^2 \Sigma \Omega \nu \tag{199}$$

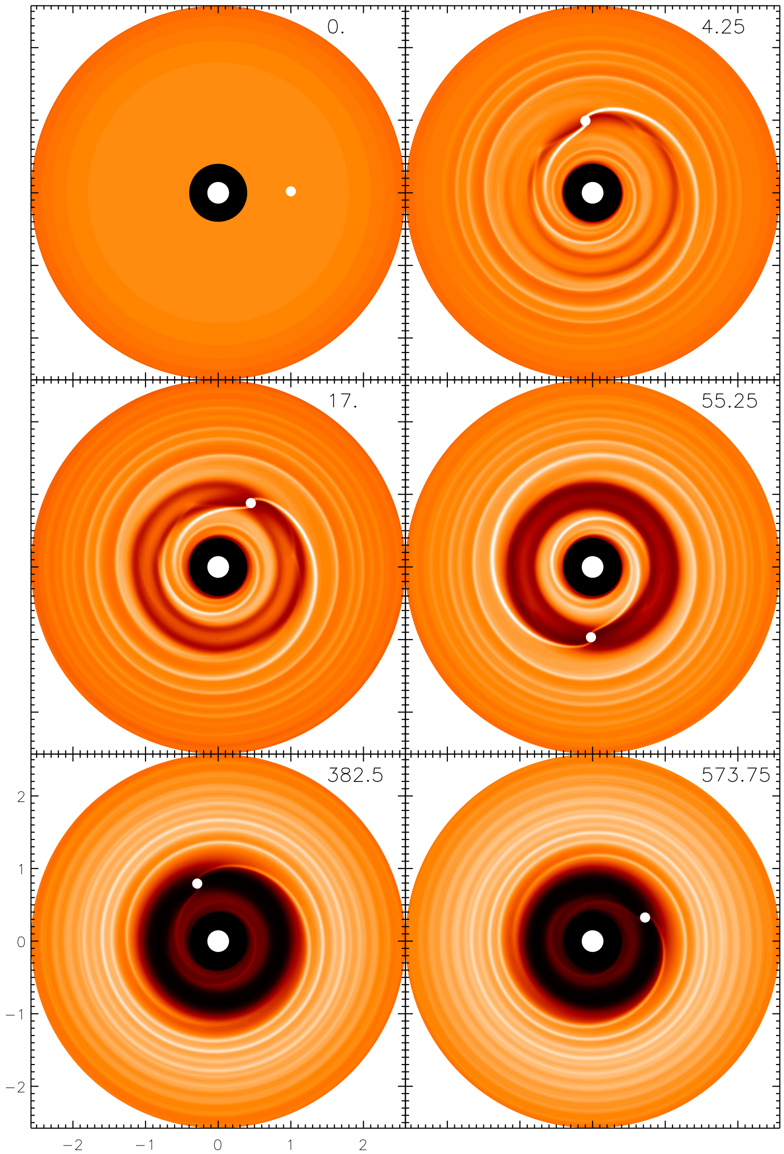
which may be written

$$\frac{m_p}{M_\odot} \gtrsim \frac{10\,\nu}{R^2\,\Omega} \tag{200}$$

For $H/R \sim 0.04$ and $\alpha = 6 \times 10^{-3}$, $\nu/(R^2\Omega) \sim 10^{-5}$ and gap formation occurs for

$$\frac{m_p}{M_\odot} \gtrsim \frac{M_J}{10}$$

(*i.e.* in the Jupiter mass range).



8.4.2 The Type II migration rate

Once a planet becomes massive enough to open a gap, orbital evolution is predicted to occur on the same local time scale as the protoplanetary disk. The radial velocity of gas in the disk is,

$$v_r = -\frac{\dot{M}}{2\pi r \Sigma}, \tag{201}$$

which for a steady disk away from the boundaries can be written as,

$$v_r = -\frac{3}{2}\frac{\nu}{r}. \tag{202}$$

If the planet enforces a rigid tidal barrier at the outer edge of the gap, then evolution of the disk will force the orbit to shrink at a rate $\dot{r}_p \simeq v_r$, provided that the local disk mass exceeds the planet mass, i.e. that $\pi r_p^2 \Sigma \gtrsim M_p$. This implies a nominal Type II migration time scale of

$$\tau_0 = \frac{2}{3\alpha} \left(\frac{h}{r} \right)_p^{-2} \Omega_p^{-1}. \tag{203}$$

For $h/r = 0.05$ and $\alpha = 10^{-2}$, the migration time scale at 5 AU is of the order of 0.5 Myr.

These estimates of the Type II migration velocity assume that once a gap has been opened, the planet maintains an impermeable tidal barrier to gas inflow. In fact, simulations show that planets are able to accrete gas via tidal streams that bridge the gap. The effect is particularly pronounced for planets only just massive enough to open a gap in the first place, so in reality planets can form gaps, migrate, and grow at the same time.